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Military Operations Research

Winter 1994

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Military Operations Research

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— *Natalie S. Addison, Publisher*

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David Rist Prize: Papers submitted in response to this call will be eligible for consideration for the ***Rist Prize.*** The committee will select the prize-winning paper from those submitted and award the prize at the 64th MORSS. If selected, the author(s) will be invited to present the paper at the 64th MORSS and to prepare it for publication in the MORS Journal, *Military Operations Research.* The cash prize is $1000. To be considered, the paper must be mailed to the MORS Office and postmarked no later than **September 29, 1995.** Please send the original, three copies and the disk.

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- Citation of related work
- Description of approach
- Statement of assumptions
- Explanation of methodology
- Analysis of data and sources
- Sensitivity of analyses (where appropriate)
- Logical development of analysis and conclusions
- Summary of presentation and results

Contribution to Military Operations Research

- Importance of problem
- Contribution to insight or solution of the problem
- Power or generality of the result
- Originality and innovation

Foundations of the General Theory of Volley Fire

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ABSTRACT

Volleyolley fire problems arise frequently in applied military operations research work. This paper develops a powerful general theory whose systematic application to volley fire problems greatly aids in their solution. The aim is to provide military analysts ready access to systematic methods whose application can greatly simplify the solution of volley fire attrition models. This paper reviews a sample of the past work on volley fire problems, develops new and more powerful methods for their solution, and illustrates their application to several volley fire situations of practical interest. In the simpler cases, the theory leads directly to elegant formulas for the expectation and variance of the number of survivors. In more complicated situations, it provides algorithms useful for numerical calculations. The theory powerfully unifies and extends previously used methods for solving volley fire problems and often provides simpler and more intuitive solutions than have previously appeared. It also yields hitherto unpublished results. Our approach also shows that volley fire models generalize many of the classical probability problems in the theory of matchings, occupancy, and statistical mechanics. In addition, it suggests potentially important new concepts, such as those for equivalent and complementary volleys. It also provides a useful system for classifying volleys into a few "canonical forms" based on their common features, which facilitate their solution by avoiding the need for *ad hoc* methods. Several potential areas for further investigation are also suggested.

INTRODUCTION

Estimating the attrition that results when a battery of weapons shoots at an array of targets is one of the most characteristic activities of military operations analysts. In many cases the shots are fired as a volley, or nearly so. In this paper a group of shots is called a volley if all of them are fired before the weapons adjust their operations on the basis of any damage done to the target array. When the state of the target array is taken to specify which targets are still alive, a more precise restatement of this definition is that the weapons act only on the state of the target array at the start of the volley, and not on any change in state that occurs while the volley is in progress.

This definition generalizes those given in the prestigious Oxford English Dictionary [1971], which defines a volley as "A simultaneous discharge of a number of firearms or artillery; a salvo," a salvo as "A simultaneous discharge of artillery or other firearms, whether with hostile intent or otherwise," and a fusillade as "A simultaneous discharge of firearms; a wholesale execution by this means." When weapons are discharged simultaneously, the battery of weapons plainly does not have time during the volley to adjust to any damage done to the target array, and in that case our definition agrees with those given in the Oxford English Dictionary. However, in this paper we continue to speak of a volley fire situation whenever the weapons do not perceive (or if for any reason they do not heed) changes in the state of the target array. In such cases, whether the shots are fired simultaneously or over an extended period of time is clearly immaterial, because at the end of the volley their effects are the same.

In addition, the results presented in this paper can be applied not only to artillery or firearms, but to a wider class of weapons including rockets, antitank weapons, intercontinental ballistic missiles, machineguns, bombs, anti-aircraft artillery, and so forth. For this reason, volleys delivered by a battery of weapons against an array of targets are often used to model attrition in military operations research. They frequently appear as components of larger models, simulations, or war games, where they are used to assess the outcomes of individual volleys, or of several successive volleys, or of countervolleys fired alternately by one side and then by the other. For one example of such component volleys, see Ketron [1983]. When successive volleys are fired, the state of the system at the end of each successive volley usually depends only on the state of the system at the start of the volley, in which case the state vector evolves according to a Markov process, the importance of which in models of combat interactions has been emphasized by Koopman [1970], among others. In this paper, several volleys of practical military interest are presented and solved to illustrate the general theory's ability to yield previously unpublished results, as well as to provide simpler and more intuitive derivations of known results.

Our theory takes for its object the deter-

mination of the outcome when a battery of weapons volleys against a target array. It makes use of techniques borrowed from the fields of combinatorics, probability, algebra, and analysis (the latter principally in connection with limit laws of probability). Although weapon batteries and target arrays consisting of a single element are technically within the purview of volley fire theory, they are normally viewed as special cases and are analyzed as duels. Thus, the theory of volley fire concentrates on cases where the battery of weapons and the target array both contain several elements. The scope of the theory is intended to include the analysis of multiple volleys by a battery against a single target array, and also exchanges of volleys where the weapons battery and the target array volley back and forth. Unfortunately, the current state of the theory does not provide a very satisfying treatment of multiple or counter volleys. Consequently, apart from a few passing remarks, this paper concentrates on single volleys with the understanding that they can be chained together in various (often *ad hoc*) ways to estimate the effects of successive volleys, if so desired.

The central object in this paper is the computation of the probability of various outcomes of a single volley. We want to know such things as:

1. How many targets can be expected to survive the volley?
2. How variable is the number of survivors?
3. What is the probability that 6 targets survive? That 12 survive? In general, what is the probability that t targets survive?
4. If the target array consists of two or more types of target, what are the correlations between the number of survivors of each type?
5. What is the probability that all targets of a specified type will be wiped out?

Until recently, volley fire problems were treated by *ad hoc* methods that gave limited results for special cases. Despite the ingenuity of some of these *ad hoc* methods, they concentrated so strongly on special cases that their general theoretical foundations tended to be hidden rather than revealed. Just very recently, it was recognized that there are some deeper and more general concepts whose systematic application to volley fire problems can greatly aid in their solution. In the simpler cases, these concepts lead directly to elegant formulas for the expectation, variance, and correlation of the number of survivors. In more complicated situations, they provide algorithms useful for numerical computations. The systematic application of these general analytical methods has led to simpler and more intuitive proofs of all of the known results in the theory and has clarified their interrelations. This approach has also led to several new and previously unknown results. This has put us in the position where, for the first time, it appears that such a thing as a theory of volley fire might exist.

It also turns out that the theory of volley fire includes as a special case all of the theory of random allocations. That familiar field of classical probability theory deals with the random allocations or distributions of r objects into n cells. Treatments of the problem of *rencontres* or random matchings; of the distribution of particles among energy states for Maxwell-Boltzman, Bose-Einstein, or Fermi-Dirac statistics in the kinetic theory of matter; and of many other famous classical problems are examples of those dealt with in the theory of random allocations. Certain aspects of this classical theory of random allocations have recently been developed extensively by the Russian mathematician Kolchin [1978] and colleagues. However, they have dealt almost exclusively with the study of limiting distributions for certain classes of random allocations. In contrast, practically all of the extant work on the theory of volley fire has been devoted to solving certain difficult combinatorial problems in the theory of probability. In the future, it may be possible and desirable to extend to the theory of volley fire some of the asymptotic results from the theory of random allocations.

Several authors have studied special cases of volley fire, and Table 1 shows a sample of the earlier work on special cases.

We will say more about these papers presently. The general theory presented in this paper supersedes these specialized approaches, because it can readily be used not only to reproduce all of the previous results, but to provide additional information not obtainable by the specialized methods. Because volley fire models have arisen in a variety of contexts, other works on them may not have come to our attention, and we apologize in advance to the authors of any volley fire analyses not listed in Table 1.

Dixon [1953] was apparently one of the first to analyze a volley fire situation. He showed by selected examples how the outcome of repeated volleys by a homogeneous battery of weapons (that is, one in which the weapons are all alike) against a homogeneous array of passive targets could be computed. He applied his results to calculate the distribution of the number of survivors for some situations in which successive waves of interceptor aircraft (the weapons battery) attack a formation of bombers that is attempting to reach its bomb release zone. Dixon's method for finding the distribution of the number of survivors at the end of a single volley requires the exhaustive enumeration of certain combinatorial configurations. This method is similar in principle to those later employed by Robertson [1956] and by Helmbold [1960]. Dixon works out a few specific examples involving no more than four weapons and only a handful of targets but does not present an explicit algorithm for enumerating the required configurations.

However, Dixon does find an important general formula for the expected number of survivors after a single volley. He argues correctly that the probability a particular weapon selects a particular target (say, target

Table 1. A Sampling of Some Early Work on Volley Fire Models

Principal author	Pub. date	Target array	Weapon battery	Allocation of fire	Provides explicit formulas for: Expected no. of survivors	Variance of no. of survivors	Distribution of no. of survivors
Dixon	1953	Homog.	Homog.	Unif. random	Yes	No	No[a]
Lavin	1953	Homog.	Homog.	Unif. random	No	No	No[a]
Wegner	1954	Homog.	Homog.	Unif. random	Yes	No	No[a]
Thomas	1956	Homog.	Homog.	Unif. random	Yes	No	Yes
Robertson	1956	Homog.	Heterog.	Unif. random	No	No	Yes[b]
Helmbold	1960	Homog.	Heterog.	Unif. random	Yes	No	Yes[b]
Rau	1964	Homog.	Homog.	Unif. random	Yes	Yes	Yes
Rau	1965	Homog.	Homog.	Unif. random	Yes	No	Yes
Ancker	1965	Homog.	Homog.	Unif. random	No	No	Yes
Helmbold	1966	Homog.	Heterog.	Unif. random	Yes	No	Yes
Helmbold	1968	Heterog.	Heterog.	Random	Yes	No	No
Karr	1974	Homog.	Homog.	Compound[c]	Yes	No	Yes
Karr	1974	Heterog.	Heterog.	Compound[c]	Yes	No	No

Notes:

a. Distributions are provided only for a few examples involving a small number of weapons and targets. A general algorithm is not explicitly stated.

b. Although a computationally well-defined algorithm for obtaining the distribution is provided, it requires as an intermediate step the cumbersome generation of certain combinatorial configurations.

c. Allocation of fire is determined by a compound process in which targets are first acquired, and then fire is allocated uniformly at random over the subarray of acquired targets.

t) is $1/T$, where T is the number of targets alive at the start of the volley. Thus, the probability that target t survives the fire of this weapon is $1 - q/T$, where q is the kill probability. Consequently, the probability that target t survives the fires of all of the W weapons in the battery is $(1 - q/T)^W$. Since this probability is the same for each target, the expected number of survivors after one volley is

$$E(T^1) = T(1 - q/T)^W.$$

This elegant result will be called "Dixon's Formula." As indicated in Table 1, it applies when a homogeneous battery of weapons volleys against a homogeneous array of passive targets, provided the weapons allocate their fire to targets selected independently and uniformly at random from the target array. Such volleys, including their generalization to volleys by a heterogeneous battery of weapons against a homogeneous target array, will be called Dixon-Robertson-Rau (DRR) volleys after three who have contributed substantially to the theory of volley fire, although in actuality none of these three provided closed form solutions for the general case where the weapons battery is heterogeneous or the allocation of fire may be nonuniform. However, such formulas are easily found using our general approach.

By using the same methods as Dixon [1953], Lavin and Wegner [1953] generate distributions of the number of survivors for additional examples of DRR volleys. They obtain expressions for the cases where $W = 1$, 2, 3, 4, and 8, and for T up to 9. They also apply the then new electronic digital computer technology to compute the matrix products required in the Markov process approach. Wegner [1954] continues in this vein and also introduces a process in which the two sides exchange volleys simultaneously.

Robertson [1956] provides an explicit algorithm for computing the distribution of survivors when a homogeneous battery of weapons volleys against a homogeneous target array and illustrates by example a method for obtaining the distribution of the number of survivors when the battery is heterogeneous. She applies the results to situations in which an infantry rifle squad (the battery) is defending its position against an assault conducted by another rifle squad. She makes no reference to the earlier work of Dixon, Lavin, and Wegner and seems to have arrived independently at her results.

Thomas [1956] derives and solves in closed form a partial difference equation for the distribution of the number of survivors in a DRR volley and obtains Dixon's formula from it. He has in mind the case of interceptors (weapons) against bombers (targets). He also presents computationally convenient recursive formulas and a generating function for the distribution of survivors, and suggests various approximations to that distribution. However, Thomas does not explicitly cite a formula for the variance of the number of survivors. He does note the Markov chain solution for successive volleys, and analyzes a volley in which each weapon in the battery may have kill probability q_1 or q_2 (with probabilities ψ_1 and $\psi_2 = 1 - \psi_1$, respectively). He also considers a heterogeneous target array consisting of just two types of target (bombers and decoys): for this case he analyzes the allocation of a fixed budget to bombers and decoys to maximize the expected number of surviving bombers.

Also in the mid 1950s Helmbold (following up ideas originated jointly by him and his colleagues Martin N. Chase, John C. Flannagan, and Hunter M. Woodall, Jr.) was examining the use of volley fire models to represent situations in which antitank weapons (the battery) are defending against tank assaults. This work was conducted in ignorance of the work of Dixon, Lavin, Wegner, Robertson, and Thomas. Some of it was later recorded in Helmbold [1960], which contains the following Generalized Dixon Formula for the case where the weapons are not all the same:

$$E\left(T^1\right) = T\prod_{w=1}^{W}\left(1 - q_w/T\right),$$

where q_w is the kill probability of weapon w. This result can be obtained by an argument similar to Dixon's but with minor modifications to allow different kill probabilities for different weapons. By the time Helmbold

[1960] was published, he had become aware of Robertson [1956], but not of Dixon [1953], Lavin [1953], Wegner [1954], or Thomas [1956].

By applying the principle of inclusion and exclusion to DRR volleys, Rau [1964] not only derives Dixon's Formula, but also finds the distribution and variance of the number of survivors. Subsequently, Rau [1965] obtained the same results by an ingenious and entirely different argument. Rau provides explicit and relatively simple formulas for the number of survivors. His formulas are much more convenient than the algorithms proposed by Robertson [1956] or Helmbold [1960]. Rau's formulas will not be repeated here, because they can be obtained by particularizing more general results given later in this paper. Rau applies his formulas to situations where surface-based air defense weapons (the battery) shoot at an intruding formation of aircraft. There is no indication in either of Rau's reports that he was aware of any of the earlier work on volley fire models.

Ancker and Williams [1965] obtain both iterative and closed form solutions for the distribution of the number of survivors for DRR volleys. They do this by setting up and solving an appropriate partial difference equation. They do not state Dixon's Formula, although it is derivable from their expressions for the distribution of the number of survivors, nor do they cite any of the works listed in Table 1. Helmbold [1966] pointed out that their argument is easily extended to obtain the distribution of survivors when the weapon battery is heterogeneous, and showed that this result leads quickly to the Generalized Dixon Formula.

Later, Helmbold [1968] further generalized these results to the case where the target array as well as the weapons battery is heterogeneous, and where, in addition, weapons select targets independently (but not necessarily uniformly) at random. Unfortunately, no results on the distribution or the variance of the number of survivors can be obtained with the methods used by Helmbold [1968]. This lack will be corrected by the results to be given later in this paper. Helmbold [1968] does not cite the earlier papers of Dixon, Lavin, Wegner, Thomas, or Rau because he was not then aware of their existence.

Karr [1974], motivated by problems in the penetration of aircraft through defended areas, considers a compound process in which each weapon of the battery independently acquires targets. After the acquisition process is completed, each weapon selects exactly one of the targets it has acquired and fires at it; however, a weapon that acquires no targets fires no shots. When the weapons battery and target array are both homogeneous, Karr derives the following formula for the expected number of survivors (we shall call it Karr's Formula, although it was originally proposed on the basis of intuition by LTG Glenn A. Kent, USAF):

$$\mathrm{E}\left(T^1\right) = T\left[1 - \left(1 - (1-d)^T\right)q/T\right]^W,$$

where d is the probability that a particular weapon will acquire a particular target and is assumed to be the same for all weapon-target pairs. Karr also gives the distribution of the number of survivors when the battery and target array are both homogeneous. He obtains the expectation of the number of survivors when the battery and array are both heterogeneous, but not its distribution or variance. Later we will show how our general theory can be used to provide that information. Karr cites none of the prior work on volley fire models.

Clearly the work just described has been disjointed, unsystematic, and failed to make the best use of earlier work. Results were usually obtained by *ad hoc* methods that (despite their other merits) had an unfortunate tendency to conceal common concepts and generally applicable principles, rather than to reveal them. The main contribution of this paper is to identify some general concepts whose systematic application to volley fire problems can greatly aid in their solution. These general concepts are natural and powerfully unify previously used methods. Several potential areas for further investigation are also suggested. The general approach developed here also reveals that volley fire models generalize in a natural way many of the classical probability problems in the theory

of matchings, occupancy, and statistical mechanics. Moreover, they yield hitherto unpublished results, and often provide simpler and more intuitive solutions than have previously appeared. In the simpler cases, these concepts lead easily and directly to elegant formulas for the expectation and variance of the number of survivors. In more complicated situations, they provide algorithms useful for numerical calculations. Applications of the approach to several volley fire situations are presented to illustrate the specific combinatorial techniques that appear most effective in analyzing volley fire problems.

That some reasonably complex problems yield easily to the new methods developed here can be illustrated by considering a sample problem in which a heterogeneous battery of weapons volleys against a heterogeneous target array, and weapons select targets independently (but not uniformly) at random. Suppose, as shown in Figure 1, that three weapons volley against an array of six targets. Weapons 1 and 3 are medium antitank weapons, and each fires three shots during the volley. Weapon 2 is a heavy antitank weapon and fires two shots during the volley. The target array consists of three medium and three heavy tanks, alternating with each other as shown in Figure 1. The heavy tank labeled as target number 4 contains the tank unit's commander. The antitank weapons are 90% reliable. Let v_{wt} be the probability that weapon w (w = 1, 2, or 3) directs a reliable shot at target t (t = 1, 2, ..., 6). Let q_{wt} be the probability that target t will be killed if a reliable shot is directed at it by weapon w, and let the numerical values of these factors be as in Table 2. Note that, in this example problem,

$$\sum_{t=1}^{6} v_{wt} = 0.90$$

for w = 1, 2, or 3 because the 90 percent reliability per shot has been included in the v_{wt} values. (The reliability factor would have the same effect on the results if, instead, it had been used to reduce the value of q_{wt}.) We assume that each weapon directs each of its shots at a target selected in accord with the v_{wt} values, but independently of all other shots fired. By employing some of the results to be presented later for volleys by independently effective weapons and a small hand calculator of the type readily available nowadays, we found in a few minutes the following information.

1. The average and standard deviation of the number of tanks surviving the volley is 3.054 and 1.007, respectively.
2. The average and standard deviation of the number of heavy tanks surviving the volley is 1.447 and 0.759, respectively.
3. The average and standard deviation of the number of medium tanks surviving the volley is 1.606 and 0.818, respectively.
4. The probability that the commander's tank survives the volley is 0.382.
5. The probability that exactly 0, 1, 2, or 3 heavy tanks survive the volley is 0.096, 0.429, 0.407, and 0.068, respectively.
6. The probability that exactly 0, 1, 2, or 3 medium tanks survive the volley is 0.082, 0.363, 0.422, and 0.133, respectively.
7. The correlation between the numbers of medium and heavy tanks surviving the volley is 0.187.

It should be emphasized that these values are *not* the result of any form of Monte Carlo simulation. Instead, they are exact values obtained by substituting the assumed values of v_{wt} and q_{wt} into exact formulas for the situation described. Consequently, they could be used to verify that a Monte Carlo simulation was operating correctly.

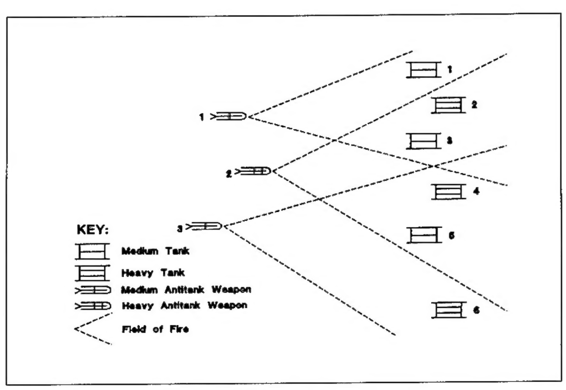

Figure 1: A Sample Volley Problem

Table 2. Values of v_{wt} and q_{wt} for the Sample Problem

	w					
	1		2		3	
t	v_{1t}	q_{1t}	v_{2t}	q_{2t}	v_{2t}	q_{1t}
1	0.23	0.70	0	N/A	0	N/A
2	0.44	0.30	0.30	0.90	0	N/A
3	0.23	0.70	0.15	0.60	0	N/A
4	0	N/A	0.30	0.90	0.35	0.30
5	0	N/A	0.15	0.60	0.20	0.70
6	0	N/A	0	N/A	0.35	0.30
No. shots	3		2		3	

2. NOTATION AND BASIC CONCEPTS

Suppose that at the start of a volley the target array consists of T targets. The state of the target array at the end of the volley will be represented by the *complexion*

$$(\tau_1, \tau_2, \ldots, \tau_T)$$

where $\tau_j = 1$ if target j is alive at the end of the volley and $\tau_j = 0$ otherwise. (A development in which additional target states are allowed is possible, but is more complex and will not be pursued here.) In principle, any properly posed question regarding the outcome of a volley can be answered if (and only if) the probabilities of each of these 2^T complexions is known. In many volley fire problems, however, a direct evaluation of the probabilities of the complexions is difficult, while the following indirect approach is more effective and in many ways more natural.

One of the characteristics of our approach is that it focuses on the probability that a target survives instead of on the probability that it is killed. This facilitates the theoretical development and yields more elegant formulas for the outcome of a volley. Consequently, we begin by defining z_j to be the event that target j survives, that is,

$$z_j = \left\{\text{complexions}\middle|\tau_j = 1\right\}.$$

The event complementary to z_j is

$$\overline{z_j} = \mathrm{I} - z_j = \left\{\text{complexions}\middle|\tau_j = 0\right\},$$

where I is the set of all complexions and so carries a probability value of unity. In this paper, set-theoretic intersections are usually written as products, so that (for example) $z_j\overline{z_k}z_m\overline{z_n}z_r$ represents the event that targets j, m, and r survive the volley while targets k and n do not.

Now consider the following *family of basic events*:

I,

z_j for $j = 1(1)T$,

z_jz_k for $j = 2(1)T$ and $k = 1(1)(j-1)$,

$z_jz_kz_l$ for $j = 3(1)T$, $k = 2(1)(j-1)$,
and $l = 1(1)(k-1)$,

...

$z_1z_2z_3 \ldots z_T$.

Here and elsewhere in this paper the notation $m = a(b)c$ denotes that m is a variable that ranges over the set of values $a, a+b, a+2b, \ldots, c-b, c$. Call a basic event that is specified by the product of exactly r z's an r-th order basic event. For each $r = 0(1)T$, there are exactly $\binom{T}{r}$ r-th order basic events, because that is the number of combinations of r z's that can be selected from the set of T z's. Consequently, there are a total of 2^T members in the family of basic events.

These basic events and their probabilities play so central a role in the general theory of volley fire that we define a volley to be *solved completely* if the probability of each basic event is known. It is well-known that the probabilities of the basic events suffice to determine the probability of any complexion, and therefore of any well-defined outcome of a volley (see Note 1). In several cases of practical interest, the basic event probabilities are easily evaluated, as will be demonstrated by the examples presented later in this paper. However, their computation inescapably requires special knowledge or assumptions regarding the tactical behavior as well as the technical military capability of the weapons and targets, and so it is not feasible to provide a useful general formula for them. For the present, we simply take for granted that all or some of the basic event probabilities for the volley in question can be

obtained.

Suppose we say that a volley is *solved to order m* if the probabilities of all basic events of order $r = 0(1)m$ are known. Many interesting and important questions can be answered easily, once a volley is solved to some low order. To illustrate this more fully, let A be an arbitrary but fixed collection or *subarray* of T_A targets, that is, subarray A consists of T_A targets "of type A." The subarray A may be identical to the full target array, or may be any proper subarray. Designate the targets in subarray A as $A_1, A_2, \ldots, A_{T_A}$. The probabilities of the following *subfamily of basic events associated with subarray A* are available whenever the volley has been solved to order T_A:

$$\begin{cases} \mathrm{I}, \\ z_{A_j} \quad \text{for } j = 1(1)T_A, \\ z_{A_j} z_{A_k} \quad \text{for } j = 2(1)T_A \text{ and } k = 1(1)(j-1), \\ \cdots \\ z_{A_1} z_{A_2} z_{A_3} \cdots z_{A_{T_A}}. \end{cases}$$

For each $r = 0(1)T_A$ this subfamily contains $\binom{T_A}{r}$ r-th order basic events, so in all there are 2^{T_A} basic events in this subfamily.

Now let $P_{A[m]}$ be the probability that exactly m targets of type A survive the volley. Then, by the principle of inclusion and exclusion as described in Feller [1950], Liu [1968], Riordan [1958], Netto [1927], Frechet [1940], Frechet [1943], Ryser [1963], and many other texts on probability and combinatorics, for $m = 0(1)T_A$

$$\begin{aligned} P_{A[m]} &= \sum_{r=m}^{T_A} (-1)^{r-m} \binom{r}{m} S_{A_r} \\ &= \sum_{r=0}^{T_A - m} (-1)^r \binom{m+r}{m} S_{A_{m+r}}, \end{aligned} \tag{1}$$

where S_{A_r} is the sum of the probabilities of all r-th order basic events in the subfamily associated with subarray A. That is, for $r = 0(1)T_A$,

$$\begin{aligned} S_{A_r} &= \sum_r P\left(z_{A_{j1}} z_{A_{j2}} \cdots z_{A_{j_r}}\right) = \\ &\sum_{j_1=r}^{T_A} \sum_{j_2=1}^{j_1-1} \sum_{j_3=1}^{j_2-1} \cdots \sum_{j_{r-1}=1}^{j_{r-2}-1} \sum_{j_r=1}^{j_{r-1}-1} P\left(z_{A_{j1}} \cdots z_{A_{j_r}}\right). \end{aligned} \tag{2}$$

When there is little chance of confusion, S_{A_r} will be referred to briefly as the *r-th order basic sum*. These basic sums appear frequently in the theory developed in this paper. Observe that the r-th order basic sum is the sum of $\binom{T_A}{r}$ basic event probabilities. Since r varies from 0 to T_A, there are exactly $T_A + 1$ basic sums. Of course, the value of the basic sum S_{A_0} is unity, since it is the probability of the basic event involving no z_{A_j} specification, that is, S_{A_0} is the probability of the set I of all complexions.

In many applications of the theory of volley fire, detailed information as to which *specific* targets survive is not essential and information regarding only the *number* of survivors is sufficient. In such cases, equation (1) shows that the problem reduces to finding the values of $T_A + 1$ basic sums, rather than the probabilities of 2^{T_A} basic events (or complexions). We proceed to show that further simplification is possible when the complete probability distribution of the number of survivors is not required, and only the values of its first few moments are needed.

Let $G_A(x)$ be the generating function for the distribution of the number of survivors, that is,

$$G_A(x) = \sum_{m=0}^{T_A} x^m P_{A[m]}. \tag{3}$$

Note that the probability that at least m targets survive can easily be generated from $G_A(x)$ — see Note 2.

Replacing $P_{A[m]}$ by its value as given by equation (1) and then interchanging the order of summation (having due regard for the region in the (m, r) plane over which the summation extends) yields

$$G_A(x) = \sum_{r=0}^{T_A} \sum_{m=0}^{r} (-1)^r S_{Ar} \binom{r}{m} (-x)^m = \sum_{r=0}^{T_A} (x-1)^r S_{Ar}. \tag{4}$$

The expectation and variance of T_A^1, the number of type A targets that survive the volley, are easily obtained from $G_A(x)$ by taking derivatives, and we find:

$$E\left(T_A^1\right) = G_A'(1) = S_{A_1} \tag{5}$$

and

$$\mathrm{Var}\left(T_A^1\right) = G_A''(1) + G_A'(1) - \left[G_A'(1)\right]^2 = 2S_{A_2} + S_{A_1} - S_{A_1}^2. \tag{6}$$

Since A may be any subarray of targets, formulas for the expectation and variance of the total number of survivors can be obtained simply by suppressing A in formulas (5) and (6). It is often important for applications that the expected number of survivors can be obtained from the first order basic sum, its variance from the first two basic sums, and (in general) its n-th order moment from the first n basic sums (see Note 3).

Now let A and B be arbitrarily prescribed subarrays containing T_A and T_B targets, respectively. Designate the targets in these subarrays as A_j, where $j = 1(1)T_A$, and as B_k, where $k = 1(1)T_B$. The subarrays A and B may overlap in any way. By definition, the correlation between the number of survivors of type A and type B is

$$\rho_{AB} = \frac{E\left(T_A^1 T_B^1\right) - E\left(T_A^1\right)E\left(T_B^1\right)}{\sqrt{\mathrm{Var}\left(T_A^1\right)\mathrm{Var}\left(T_B^1\right)}}. \tag{7}$$

The variance and expectation of T_A^1 and T_B^1 in (7) can be found from equations (5) and (6). To find $E\left(T_A^1 T_B^1\right)$, observe that

$$T_A^1 = \sum_{j=1}^{T_A} \tau_{A_j},$$

and similarly for T_B^1, so that

$$T_A^1 T_B^1 = \sum_{j=1}^{T_A} \sum_{k=1}^{T_B} \tau_{A_j} \tau_{B_k}.$$

By well-known properties of the indicator functions and expectations, it then follows that

$$E\left(T_A^1 T_B^1\right) = \sum_{j=1}^{T_A} \sum_{k=1}^{T_B} E\left(\tau_{A_j} \tau_{B_k}\right) = \sum_{j=1}^{T_A} \sum_{k=1}^{T_B} P\left(z_{A_j} z_{B_k}\right), \tag{8}$$

and we see that all of the quantities appearing in equation (7) are available whenever the volley has been solved to the second order.

The preceding development can easily be extended to obtain expressions for the expectation, variance, and correlation between weighted survivor functions, such as

$$M_A^1 = M_{A_0} + \sum_{j=1}^{T_A} M_{A_j} \tau_j, \tag{9}$$

where the constant term M_{A_0} is present only when non-zero weights are assigned to losses (see Note 4). This slight generalization can be treated by a straightforward extension of the methods used to analyze the special case in which $M_{A_0} = 0$ and $M_{A_j} = 1$ for $j = 1(1)T_A$, to which we now return.

3. EQUIVALENT VOLLEYS AND CANONICAL FORMS

Volleys frequently are described by specifying how targets are to be acquired, how fire is to be allocated among the acquired targets, and how the damage done to the target array by various allocations of fire is to be determined. For many purposes, such descriptions are absolutely essential.

On the other hand, the basic concepts introduced in the preceding section make no reference to the verbal description of a volley. Instead, they deal only with its basic event probabilities. When these basic event probabilities are obtained for a number of different

volleys, it is observed that the mathematical expressions for them sometimes exhibit the same functional form. Recognition of this common functional form can be of capital importance, since all volleys whose basic event probabilities have the same functional form can be analyzed by the same mathematical methods. When the basic event probabilities for two volleys can be put in the same functional form, the volleys are said to be *equivalent*, and the common functional form is said to be their *canonical form*. We will not here attempt to formalize exactly when two mathematical expressions can be put into the same functional form. Instead, we go directly to examples of canonical forms, each of which is analyzed more fully later in this paper.

3-1. Independently Survivable Targets. We say that the targets within a subarray A of T_A targets are independently survivable if the z_{A_j}'s for $j = 1(1)T_A$ are independent events. In that case, the canonical form for the basic event probabilities is

$$P\left(z_{A_{j1}} z_{A_{j2}} \cdots z_{A_{jr}}\right) = \prod_{n=1}^{r} P\left(z_{A_{jn}}\right), \quad (10)$$

where the argument on the left is any r-th order basic event associated with subarray A. All volleys which are equivalent to a volley of this form are called volleys against independently survivable targets.

Some of the properties shared by all such volleys are as follows. All targets in the full array are independently survivable if, and only if, they are independently survivable within every subarray. Moreover, such a volley can be solved completely whenever it can be solved to the first order, since all of its basic event probabilities are known functions of the first order basic event probabilities.

3-2. Exchangeably Survivable Targets. The targets within a subarray A of T_A targets are called exchangeably survivable if the z_{A_j}'s for $j = 1(1)T_A$ are exchangeable events, that is, if the probability of any basic event in the subfamily of basic events associated with subarray A depends only on the *number* of z_{A_j}'s in its specification, but not on which *particular* z_{A_j}'s appear in it. Specifically, targets are exchangeably survivable within a subarray A whenever

$$P\left(z_{A_j}\right) = P\left(z_{A_1}\right) = P_{A1} \quad \text{for } j = 1(1)T_A,$$

$$P\left(z_{A_j} z_{A_k}\right) = P\left(z_{A_1} z_{A_2}\right) = P_{A2} \quad \text{for } j = 2(1)T_A \text{ and } k = 1(1)(j-1),$$

and so forth. Thus, the canonical form for a volley against exchangeably survivable targets is

$$P\left(z_{A_{j1}} z_{A_{j2}} \cdots z_{A_{jr}}\right) = P\left(z_{A_1} z_{A_2} \cdots z_{A_r}\right) = P_{Ar}, \quad (11)$$

where $j_1 = r(1)T_A$ and $j_n = 1(1)(j_{n-1} - 1)$ for $n = 2(1)r$. The concept of exchangeability and some of its connections with other topics in the theory of probability and mathematical statistics can be found in Feller [1966], Loeve [1960], De Finetti [1974], and Frechet [1940], among others.

Some of the properties possessed by all volleys against exchangeably survivable targets are as follows. If all targets in the full array are exchangeably survivable, then they are exchangeably survivable within any subarray. Such a volley is solved completely once each of the T values P_r for $r = 1(1)T$ are known, where P_r is the probability of the r-th order basic event $z_1 z_2 \ldots z_r$.

In addition, when the targets in a subarray A of T_A targets are exchangeably survivable the general equations (1) through (6) immediately reduce to the following elegant forms:

$$S_{A_r} = \binom{T_A}{r} P_{Ar}, \quad (12)$$

$$P_{A[m]} = \binom{T_A}{m} \sum_{r=0}^{T_A - m} (-1)^r \binom{T_A - m}{r} P_{A(m+r)} = \binom{T_A}{m} \sum_{r=m}^{T_A} (-1)^{m+r} \binom{T_A - m}{r - m} P_{Ar}, \quad (13)$$

$$G_A(x) = \sum_{r=0}^{T_A} (x-1)^r \binom{T_A}{r} P_{Ar}, \quad (14)$$

$$E\left(T_A^1\right) = T_A P_{A1}, \text{ and} \tag{15}$$

$$\begin{aligned}&\mathrm{Var}\left(T_A^1\right)\\ &= T_A(T_A - 1)P_{A2} - T_A P_{A1}(T_A P_{A1} - 1).\end{aligned} \tag{16}$$

When A and B are subarrays of T_A and T_B targets, and if the targets are exchangeably survivable within their union subarray, $A \cup B$, then it can be shown that

$$\begin{aligned}E\left(T_A^1 T_B^1\right) &= \sum_{j=1}^{T_A} \sum_{k=1}^{T_B} P\left(z_{A_j} z_{B_k}\right)\\ &= T_A T_B P_{A2} + T_{A\cap B}(P_{A1} - P_{A2}),\end{aligned}$$

so that

$$\rho_{AB} = \frac{P_{A2}(T_A T_B - T_{A\cap B}) - (T_A T_B P_{A1} - T_{A\cap B})P_{A1}}{\sqrt{\mathrm{Var}\left(T_A^1\right)\mathrm{Var}\left(T_B^1\right)}} \tag{17}$$

where $T_{A\cap B}$ is the number of targets in both A and B, that is, in $A \cap B$.

3-3. Independently versus Synergistically Effective Weapons. Suppose that a battery of W weapons volleys against an array of T targets. Suppose that we know the basic event probabilities when each of the W weapons acts alone and all other weapons are silent. Let $p_w(z_j)$, $p_w(z_j z_k)$, and so forth, be the basic event probabilities for a volley by weapon w acting alone against the target array. Then the canonical form for a volley by independently effective weapons is

$$P\left(z_j\right) = \prod_{w=1}^{W} p_w\left(z_j\right),$$

$$P\left(z_j z_k\right) = \prod_{w=1}^{W} p_w\left(z_j z_k\right), \tag{18}$$

and so forth for each basic event probability. Consequently, volleys by independently effective weapons can be analyzed by temporarily setting aside all but one of the weapons in the battery, solving each of the resulting single weapon volleys, and recombining them via the independence of their individual effects. The volley used as an example in the Introduction is a volley by a battery of independently effective weapons.

If the weapons in a battery are not independently effective, then we say that they are *synergistically effective*. A volley by synergistically effective weapons cannot be solved completely by analyzing only its single weapon subvolleys.

Observe that a volley by independently effective weapons is against an array of independently (respectively, exchangeably) survivable targets whenever each of its single weapon subvolleys is against an array of independently (respectively, exchangeably) survivable targets.

3-4. Independently Effective Point Fire Weapons and Munitions. Suppose that a battery of W independently effective weapons volleys against an array of T targets, and consider

$$\begin{aligned}P\left(z_j z_k \cdots z_m\right) &= \prod_{w=1}^{W} p_w\left(z_j z_k \cdots z_m\right)\\ &= \prod_{w=1}^{W}\left[1 - p_w\left(\overline{z_j z_k \cdots z_m}\right)\right]\\ &= \prod_{w=1}^{W}\left[1 - p_w\left(\overline{z_j} \cup \overline{z_k} \cup \cdots \cup \overline{z_m}\right)\right],\end{aligned} \tag{19}$$

where $\cup$ indicates the set theoretic union of events. Now, in some volleys, a weapon may be unable to kill more than one target. A weapon that is unable to kill more than one target per volley will be called a point fire weapon. For each such weapon,

$$\begin{aligned}p_w\left(\overline{z_j}\,\overline{z_k}\right) &\equiv p_w\left(\overline{z_j}\,\overline{z_k}\,\overline{z_r}\right)\\ &\equiv \cdots \equiv p_w\left(\overline{z_j}\,\overline{z_k} \cdots \overline{z_m}\right) \equiv 0,\end{aligned}$$

so that

$$\begin{aligned}&p_w\left(\overline{z_j} \cup \overline{z_k} \cup \cdots \cup \overline{z_m}\right)\\ &\equiv p_w\left(\overline{z_j}\right) + p_w\left(\overline{z_k}\right) + \cdots + p_w\left(\overline{z_m}\right).\end{aligned} \tag{20}$$

Consequently, for volleys by batteries composed exclusively of independently effective point fire weapons, the canonical form is

$$P\left(z_{j_1} z_{j_2} \cdots z_{j_r}\right) = \prod_{w=1}^{W} \left\{1 - \sum_{n=1}^{r} p_w\left(\overline{z_{j_n}}\right)\right\}. \quad (21)$$

A volley by a battery of independently effective point fire weapons can be solved completely by finding each of the WT values $p_w\left(\overline{z_j}\right)$ for $w = 1(1)W$ and $j = 1(1)T$, where $p_w\left(\overline{z_j}\right)$ is the probability that target j is killed during the subvolley in which weapon w acts alone against the full target array.

A weapon that is not a point fire weapon will be called an *area fire weapon.* Equations (18) or (19) give the canonical form for a volley by a battery of independently effective area fire weapons.

The definition of a point fire weapon needs to be broadened slightly to accommodate comfortably a number of important applications. For example, suppose that one of the weapons is a rifle that in the course of a volley may fire a number of shots and kill several targets. Under the definition given above, the rifle fails to qualify as a point fire weapon, although both common sense and conventional military terminology agree in ascribing "point fire" qualities to rifles. One appropriate response to this situation is to introduce the concept of independently effective *point fire munitions,* as follows.

Suppose that each weapon in a volley fires a certain number of shots. Let S_w be the number of shots fired by weapon w. Let $p_{ws}(z_j)$, $p_{ws}(z_j z_k)$, and so forth, be the basic event probabilities for a "volley" consisting of just shot number s from weapon w acting alone against the full target array. We say that weapon w fires independently effective munitions if

$$p_w\left(z_j\right) = \prod_{s=1}^{S_w} p_{ws}\left(z_j\right),$$

$$p_w\left(z_j z_k\right) = \prod_{s=1}^{S_w} p_{ws}\left(z_j z_k\right), \quad (22)$$

and so on. If, in addition, shot s can kill at most one target, so that

$$p_{ws}\left(\overline{z_j}\,\overline{z_k}\right) \equiv p_{ws}\left(\overline{z_j}\,\overline{z_k}\,\overline{z_r}\right) \equiv \cdots$$
$$\equiv p_{ws}\left(\overline{z_j}\,\overline{z_k} \cdots \overline{z_m}\right) \equiv 0,$$

then we say that shot s from weapon w is a point fire munition. In that case,

$$p_{ws}\left(z_{j_1} z_{j_2} \cdots z_{j_r}\right) = 1 - \sum_{n=1}^{r} p_{ws}\left(\overline{z_{j_n}}\right). \quad (23)$$

If all the shots fired by weapon w are independently effective point fire munitions, then

$$p_w\left(z_{j_1} z_{j_2} \cdots z_{j_r}\right) = \prod_{s=1}^{S_w} \left\{1 - \sum_{n=1}^{r} p_{ws}\left(\overline{z_{j_n}}\right)\right\}. \quad (24)$$

If, in addition, the shots fired by the various weapons in the battery are independently effective, then so are the weapons. In that case, the canonical form for a volley of independently effective point fire munitions will be written as

$$P\left(z_{j_1} z_{j_2} \cdots z_{j_r}\right) = \prod_{w=1}^{W} \prod_{s=1}^{S_w} \left\{1 - \sum_{n=1}^{r} p_{ws}\left(\overline{z_{j_n}}\right)\right\}. \quad (25)$$

Observe that a volley of independently effective point fire munitions is equivalent to a volley delivered by a battery of

$$W^0 = \sum_{w=1}^{W} S_w \quad (26)$$

independently effective point fire weapons, each of which fires exactly one shot. The equivalence is obtained by replacing the original battery of W weapons by the battery of W^0 weapons, and arranging things so that their kill probabilities correspond to those of the shots in the original volley. Therefore, in the theoretical treatment, we may freely replace a volley of independently effective point fire munitions by an equivalent volley of independently effective point fire weapons.

3-5. Summary of Canonical Forms. The forego-

ing suggests the taxonomy of canonical forms shown in Table 3. Each block in this table represents a canonical form possessing the combination of target and weapon attributes indicated by the column and row. The named volleys listed in the blocks of Table 3 are examples or special cases of canonical forms for the block. Volleys of independently effective point fire munitions are listed as if they were replaced by an equivalent volley of independently effective point fire weapons. Each of the examples listed in Table 3 is described at length and solved completely in subsequent sections of this paper. If no examples are listed in a block, it indicates that we are not aware of any practically useful examples of that canonical form which can be solved completely. (In fact, the Bellwether Volley was contrived to provide a solvable example of a volley by synergistically effective weapons against an array of targets that are neither independently nor exchangeably survivable, rather than for its practical utility.) It appears that describing and solving volleys that not only fit the characteristics indicated by the lower right hand blocks of Table 3, but that also have a spectrum of valuable applications, is a worthwhile area of research.

4. VOLLEYS AGAINST INDEPENDENTLY SURVIVABLE TARGETS

These are the easiest volleys to analyze. Indeed, it must be admitted that the results in this section are well known and are traditionally obtained by elementary probability arguments that are simpler and more direct than those based on the general theory of volley fire. That they can also be derived using the general machinery developed above illustrates its application to simple situations, as well as to more complicated ones where elementary probability arguments do not apply. It also demonstrates that the results obtained using the general methods do indeed agree with those reached by more familiar approaches. It will be found that the general methods are more precise and rigorous than the usual informal arguments. Moreover, the results are used later in this paper.

Recall that the canonical form for a volley against an array of T independently survivable targets is (*cf.* equation (10))

$$P\left(z_{j_1} z_{j_2} \cdots z_{j_r}\right) = \prod_{n=1}^{r} P\left(z_{j_n}\right).$$

Table 3. Taxonomy of Canonical Forms

Weapons	Targets: Independently survivable	Targets: Exchangeably survivable	Targets: Neither independently nor exchangeably survivable
Independently effective point fire	Gauntlet Volley* ICBM Volley*	Dixon-Robertson-Rau (DRR) Volley	Helmbold Volley* Burst Fire Volley* Hide-and-seek Volley
Independently effective area fire		Multishot Karr Volley	
Synergistically effective	Artillery Volley	Redundantly Survivable Target Volley	Bellwether Volley

* Since this is a volley of independently effective point fire munitions, or for other reasons is equivalent to a volley by independently effective point fire weapons, it is listed in that category.

Obviously, the targets within any subarray are also independently survivable. Helmbold [1992] proves the following:

Theorem 1: Let V be a volley against independently survivable targets and let A be any subarray of T_A targets. Then the generating function for the distribution of the number of type A survivors is

$$G_A(x) = \prod_{j=1}^{T_A}\left[1+(x-1)P\left(z_{A_j}\right)\right]. \qquad (27)$$

Corollary 1.1: Let V be a volley against independently survivable targets. Let A and B be disjoint subarrays of T_A and T_B targets, respectively. Let $A \cup B$ be the subarray consisting of the type A and type B targets. Then the following statements are true.

(i) $G_{A\cup B}(x) = G_A G_B$,

(ii) $\rho_{AB} = 0$,

(iii) $E\left(T^1_{A\cup B}\right) = E\left(T^1_A\right) + E\left(T^1_B\right)$,

(iv) $\mathrm{Var}\left(T^1_{A\cup B}\right) = \mathrm{Var}\left(T^1_A\right) + \mathrm{Var}\left(T^1_B\right)$, and

(v) $P_{A\cup B[m]} = \sum_{k=0}^{m} P_{A[m-k]} P_{B[k]}$.

Corollary 1.2: Let V be a volley against independently survivable targets and let A be any subarray of T_A targets. Then

$$E\left(T^1_A\right) = \sum_{j=1}^{T_A} P\left(z_{A_j}\right)$$

and

$$\mathrm{Var}\left(T^1_A\right) = \sum_{j=1}^{T_A} P\left(z_{A_j}\right)\left\{1-P\left(z_{A_j}\right)\right\}.$$

Corollary 1.3: Let V be a volley against independently survivable targets. Suppose that the targets in subarray A are exchangeably survivable. Then the following assertions are true:

(i) The probability that exactly m of the type A targets survive is

$$P_{A[m]} = \binom{T_A}{m} P_A^m (1-P_A)^{T_A-m}$$

where

$$P_A = P(z_{A1}) \equiv P(z_{Am}) \text{ for } m = 1(1)T_A$$

is the survival probability of an arbitrarily selected target of type A.

(ii) $E\left(T^1_A\right) = T_A P_A$.

(iii) $\mathrm{Var}\left(T^1\right) = T_A P_A (1-P_A)$.

Now we will consider some particular cases of volleys against independently survivable targets. Here, as in the discussion of other canonical forms, these examples illustrate the connection between the somewhat lifeless abstract canonical form of a volley and the animated, often colorful applied versions familiar to military operations analysts. In describing particular volleys, it is usually helpful to think of the battery of weapons as going in turn through the phases of target acquisition, allocation of fire, and achievement of effects. In the acquisition phase, candidates for attack by one or more weapons are obtained from the target array. In the allocation phase, fire from the weapons is allocated to the acquired targets. In the effects phase, the damage done by the allocated fire is determined. With this concept of the volley process in mind, we turn to the example of the Gauntlet Volley.

4-1. The Gauntlet Volley. The informal mental image of the action in a Gauntlet Volley is that each target separately "runs the gauntlet," that is, it faces and is subject to attack by each of the weapons in turn, with each weapon-target combination encounter being a separate engagement. Alternatively, we may think of each weapon as moving in turn from one target to the next, singlehandedly engaging each target it comes to. Whichever intuitive picture is used, a Gauntlet Volley may be defined by the following postulates.

G-1: The probability that weapon w acquires target t is $a_w(t)$, and is independent of other acquisitions.

G-2: Each weapon may fire up to T shots at the target array, depending on how many targets it acquires. The probability that weapon w allocates one shot to target t is $v_w(t)$ if w acquires target t, and is zero otherwise, independent of what other events occur during the volley.

G-3: The probability that target t is killed by weapon w is $q_w(t)$ if a shot from weapon w is allocated to target t, and zero otherwise, independent of what other events occur during the volley.

The Gauntlet Volley is easily solved by observing that it is a volley against an array of independently survivable targets in which

$$P(z_t) = \prod_{r=1}^{W} \{1 - a_w(t) v_s(t) q_w(t)\},$$

so the results of Theorem 1 and its corollaries apply. The form of $P(z_t)$ shows that a Gauntlet Volley is also a volley by a battery of independently effective weapons. In general, the weapons of a Gauntlet Volley can kill more than one target, and so are area fire weapons. However, by postulate G-3, the munitions are independently effective point fire munitions, so a Gauntlet Volley is equivalent to a volley by point fire weapons. This justifies the location of the Gauntlet Volley entry in Table 3.

4-2. The ICBM Volley. The name of this volley was chosen because it has frequently been used to obtain quick estimates of the effects of a salvo of intercontinental ballistic missiles. It satisfies the following postulates.

ICBM-1: Each weapon acquires all of the targets in the target array.

ICBM-2: Each weapon fires exactly one shot. Shots are allocated as evenly as possible to the targets. More precisely, let $[W/T]$ be the greatest integer not larger than W/T, and let $R(W, T) = W - T[W/T]$ be the remainder when W is divided by T. Then $[W/T] + 1$ shots are allocated to each of the first $R(W, T)$ targets and $[W/T]$ shots are allocated to each of the remaining $T - R(W, T)$ targets.

ICBM-3: The probability that target t is killed by the shot from weapon w is $q(t)$ if weapon w's shot is allocated to target t, and zero otherwise, independent of what other events occur during the volley.

It is easily seen that an ICBM Volley is equivalent to the Gauntlet Volley V^0 in which a single weapon volleys against the target array, and in which the acquisition probabilities are $a^0(t) = 1$ for $t = 1(1)T$, the allocation probabilities are $v^0(t) = 1$ for $t = 1(1)T$, and the conditional kill probabilities are

$$q^0(t) = \begin{cases} 1 - \{1 - q(t)\}^{[W/T]+1}, & \text{for } t = 1(1)R(W,T) \\ 1 - \{1 - q(t)\}^{[W/T]}, & \text{for } t = (R(W,T)+1)(1)T. \end{cases}$$

This equivalence justifies the location of the ICBM Volley entry in Table 3.

When the target array is partitioned into two distinct subarrays A and B such that A contains the first $R(W, T)$ targets, then it is easily seen from the equivalent Gauntlet Volley V^0 that

$$P(z_t) = \{1 - q(t)\}^{[W/T]+1} \quad \text{for } t \in A, \text{ and}$$

$$P(z_t) = \{1 - q(t)\}^{[W/T]} \quad \text{for } t \in B.$$

Corollary 1.1 applies to yield

$$E(T^1) = \sum_{t=1}^{R(W,T)} \{1 - q(t)\}^{[W/T]+1} + \sum_{t=R(W,T)+1}^{T} \{1 - q(t)\}^{[W/T]}$$

$$\rho_{AB} = 0,$$

and so forth.

Now suppose that $q(t) = q_A$ for all $t \in A$ and $q(t) = q_B$ for all $t \in B$, and let

$$P_A = (1 - q_A)^{[W/T]+1}, \text{ and}$$

$$P_B = (1 - q_B)^{[W/T]}.$$

Then Corollaries 1.1 and 1.3 apply to yield the familiar formulas

$$E\left(T^1\right) = R(W,T)P_A + \{T - R(W,T)\}P_B,$$

$$\operatorname{Var}\left(T^1\right) =$$

$$R(W,T)P_A(1-P_A) + \{T - R(W,T)\}P_B(1-P_B),$$

$$P_{A[m]} = \binom{R(W,T)}{m} P_A^m (1-P_A)^{R(W,T)-m}$$

for $m = 0(1)R(W,T)$,

$$P_{B[m]} = \binom{T - R(W,T)}{m} P_B^m (1-P_B)^{T-R(W,T)-m}$$

for $m = 0(1)\{T - R(W,T)\}$, and

$$P_{[m]} = \sum_{k=0}^{m} P_{A[m-k]} P_{B[k]} \quad \text{for } m = 0(1)T.$$

4-3. The Artillery Volley. The Artillery Volley is often used to estimate the effects of fragmenting ordnance delivered by artillery, aircraft, mortars, rockets, and so forth. It may be described as follows.

A-1: Individual targets *per se* are not acquired. However, an area believed to contain targets is acquired.

A-2: There are W weapons. Weapon w fires S_w shots. Shots are not allocated to individual targets, but are allocated stochastically to particular ground zeros in such a way that $\sigma_{ws}(u,v)dudv$ is the probability that shot s from weapon w has its ground zero located almost exactly at the point (u, v). Each ground zero distribution $\sigma_{ws}(u, v)$ is independent of the actual ground zeros of other shots.

A-3: The probability that shot s from weapon w kills target t when target t is located almost exactly at (x, y) and the shot's ground zero is located almost exactly at (u, v) is given by the damage function $D_{wst}(x-u, y-v)$. Shots are independently effective given their ground zeros, that is, the probability that target t survives all shots when it is located almost exactly at (x, y) and the ground zero of shot s from weapon w is located almost exactly at (u_{ws}, v_{ws}) is equal to

$$F_t = \prod_{w=1}^{W} \prod_{s=1}^{S_w} \left\{1 - D_{wst}\left(x - u_{ws}, y - v_{ws}\right)\right\}.$$

A-4: The probability that target t is located almost exactly at (x, y) is $p_t(x, y)dxdy$, independently of the locations of other targets and of the ground zeros of the shots.

Observe that, by virtue of the above postulates, an Artillery Volley is a volley against an array of independently survivable targets. Neither the weapons nor the munitions are point fire. As explained in Note 5, the munitions are independently effective only conditionally on their ground zeros, that is, in the sense specified in postulate A-3, and do not conform to the definition of independently effective munitions in the sense expressed by equation (18). These observations justify the location of the Artillery Volley entry in Table 3. Schroeter [1984] also develops expressions for the expectation and higher moments of T^1, the number of targets that survive an artillery volley.

5. VOLLEYS AGAINST EXCHANGEABLY SURVIVABLE TARGETS

The canonical form of a volley against exchangeably survivable targets is given by equation (11). It is clear that if all the targets in an array are exchangeably survivable, so are the targets in any subarray. When the targets are exchangeably survivable, the elegant formulas (12) through (17) apply.

In general, the state space for a volley consists of the 2^T possible complexions of the target array, and the sequence of complexions generated as successive volleys are fired is a Markov chain with 2^T states. However, when the targets are exchangeably survivable, only the number of survivors matters (that is, only

those complexions which differ in the number of survivors are distinguishable), and the sequence of the number of survivors generated as successive volleys are fired is a Markov chain with only $T + 1$ states. The transition probabilities for the latter Markov chain are given by $P_{[m]}$, which may be calculated using equation (13). Consequently, volleys against exchangeably survivable targets are much easier to analyze than are volleys against arrays of targets that are neither independently nor exchangeably survivable. Particular examples of volleys against exchangeably survivable targets are given later in this paper (the names of these volleys are shown in Table 3).

6. VOLLEYS BY INDEPENDENTLY EFFECTIVE POINT FIRE WEAPONS OR MUNITIONS

In this section we show that the general theory developed earlier easily yields results for volleys by independently effective point fire weapons that are difficult to derive by other methods. In fact, we show that application of the general theory allows us to develop new results for some of these volleys. Recall that the canonical forms for volleys by batteries of independently effective weapons or munitions are given by equations (18) or (22), respectively.

6-1. The Dixon-Robertson-Rau Volley. The Dixon-Robertson-Rau (DRR) Volley occurs quite frequently in applications and also serves as a prototype for the study of other volleys by independently effective point fire weapons because it can be analyzed in considerable detail and has intuitively appealing solutions. It may be defined by the following postulates.

DRR-1: Each weapon in a battery of W weapons acquires all of the T targets in the array.

DRR-2: Each weapon fires exactly one shot, which it allocates to a target selected uniformly and independently at random from the target array. (That is, the probability that weapon w directs its shot at target t is equal to $1/T$ and is independent of the other events that occur during the volley.)

DRR-3: The probability that target t is killed by the shot from weapon w is q_w if w allocates its shot to t, and zero otherwise, independent of the other events that occur during the volley.

Observe that the DRR volley is a volley by independently effective point fire weapons, in which

$$p_w(\overline{z_t}) =$$

$$\sum_{j=1}^{T} \Pr(\text{weapon } w \text{ kills target t} | w \text{ fires at target } t)$$

$$\times \Pr(w \text{ fires at target } j)$$

$$= q_w/T.$$

Because $p_w(\overline{z_t})$ is independent of t, the targets in a DRR volley are exchangeably survivable. Then, for any subarray A of T_A targets, equations (21) and (11) show that

$$P_{Ar} = \prod_{w=1}^{W}(1 - rq_w/T) \quad \text{for } r = 1(1)T_A.$$

Therefore, in general, $P_{A2} \neq P_{A1}^2$, and hence a DRR Volley generally is *not* a volley against independently survivable targets. These observations justify the location of the DRR Volley entry in Table 3.

Because equations (12) through (17) apply, we obtain immediately

$$S_{Ar} = \binom{T_A}{r}\prod_{w=1}^{W}(1 - rq_w/T) \quad \text{for } r = 0(1)T_A,$$

$$E(T_A^1) = T_A \prod_{w=1}^{W}(1 - q_w/T),$$

$$\mathrm{Var}(T_A^1) = T_A(T_A - 1)\prod_{w=1}^{W}(1 - 2q_w/T)$$
$$+ E(T_A^1)\{1 - E(T_A^1)\},$$

$P_{A[m]} =$

$$\binom{T_A}{m}\sum_{r=0}^{T_A}(-1)^r\binom{T_A - m}{r}\prod_{w=1}^{W}\left(1-(m+r)q_w/T\right),$$

and so forth. Observe that, when $q_w = 1$ for $w = 1(1)W$, the formulas for the DRR Volley give the expectation and variance of the number of empty cells in the classical occupancy problem, and $P_{A[m]}$ gives the probability that exactly m cells of an arbitrarily chosen collection of T_A cells are empty (compare this observation to Feller [1950]). (See also Note 6.)

Clearly, a volley in which there is but one weapon that fires a total of W shots, where each shot is allocated to a target independently at random from the target array and the w-th shot has kill probability q_w, is equivalent to a DRR volley. In fact, the number of weapons and the number of shots per weapon can be changed at will, subject only to the conditions that a total of W shots be fired, that each of the shots be allocated to a target selected independently at random from the target array, and that the kill probabilities of the shots correspond one-to-one with the q_w for $w = 1(1)W$. For example, a volley by W^0 weapons, each of which fires S shots with each shot fired at a target selected independently at random from the target array, in which the kill probability gradually improves on each shot, so that

$$q_{w^0 1} \le q_{w^0 2} \le \ldots q_{w^0 S}$$

for each $w^0 = 1(1)W^0$, is equivalent to a DRR volley by a battery of $W = W^0 S$ weapons in which

$$q_w = \begin{cases} q_{w1} & \text{for } w = 1(1)W^0 \\ q_{w-W^0,2} & \text{for } w = \left(W^0+1\right)(1)\left(2W^0\right) \\ \vdots & \\ q_{w-(S-1)W^0,S} & \text{for } w = \left[(S-1)W^0+1\right](1)\left(SW^0\right). \end{cases}$$

(The assumed increase in kill probability is, of course, not essential. The important conditions are that the shots be independently effective, that each of them be directed at a randomly selected target, and that the targets are all alike.)

6-2. The Helmbold Volley. The DRR Volley can be generalized considerably at slight effort. For example, suppose a Helmbold Volley is defined by the following postulates.

H-1: Each weapon acquires all T targets in the target array.

H-2: Weapon w fires S_w shots during the volley, allocated at the rate of one target per shot. The probability that shot s from weapon w is allocated to target t is $v_{ws}(t)$ and is independent of the allocations made on other shots by the same or any other weapon.

H-3: The probability that shot s from weapon w kills target t is $q_{ws}(t)$ if that shot is allocated to target t, and is zero otherwise, independent of the other events that occur during the volley.

This obviously is a volley of independently effective point fire munitions, so that equation (25) applies with $p_{ws}\left(\overline{z_t}\right) = v_{ws}(t)q_{ws}(t)$. Hence, when A is any subarray of T_A targets,

$$S_{A1} = \sum_{j=1}^{T_A}\prod_{w=1}^{W}\prod_{s=1}^{S_w}\left\{1 - v_{ws}\left(A_j\right)q_{ws}\left(A_j\right)\right\}, \text{ and}$$

$$S_{A2} = \sum_{j=2}^{T_A}\sum_{k=1}^{j-1}\prod_{w=1}^{W}\prod_{s=1}^{S_w}$$

$$\left\{1 - v_{ws}\left(A_j\right)q_{ws}\left(A_j\right) - v_{ws}\left(A_k\right)q_{ws}\left(A_k\right)\right\}.$$

The antitank weapon volley presented in the introductory section is an example of a Helmbold Volley, and the equations above and others (such as equations (1), (5), (6), (7), and (8)) were used to calculate the numerical values for it.

Although Helmbold Volleys are volleys of independently effective point fire muni-

tions, it is clear that in general they are not volleys against exchangeably survivable targets. This justifies the location of the Helmbold Volley in Table 3.

In a Helmbold Volley it is not necessary that $\sum_{t=1}^{T} v_{ws}(t) = 1$, that is, it is not required that each shot be allocated to some target. Allocation probabilities that do not sum to unity may be interpreted as indicating that some shots are allocated to false targets (either deliberately deceptive "dummies," or inadvertent spurious targets), or that some shots are lost as the result of malfunctions, duds, and/or human error, or in other ways.

Observe that the availability of explicit solutions to volleys such as the Helmbold Volley and others smooths the way to an investigation of various optimization problems in connection with volley fire problems. For example, tactical problems such as the best arrangement of overlapping fields of fire or the value of trading rate of fire for improved accuracy can be investigated using the expected number of survivors as an objective function that is to be minimized. Some force structure issues could be clarified by evaluating the impact of different small unit organizations on the number of survivors, and so forth.

If the targets and weapons are all different, so that the $v_{wt}q_{wt}$ products are all different, then the evaluation of S_r involves $\binom{T}{r}$ terms in its summation. So if there are many targets (say 100 targets), then to evaluate S_{50} involves the summation of $\binom{100}{50} \cong 168{,}000$ moles of terms, each term being a product of W factors, each factor being of the form $1 - v_{w1}q_{w1} - \dots - v_{w50}q_{w50}$. So when there are 100 targets, it is not practical to solve the volley *completely* by evaluating all of the basic sums. Fortunately, for most practical purposes, solving the volley completely is not really necessary, and we can make do with just the first few moments of the distribution of the number of survivors, such as the expectation and variance. For 100 targets, the expected value can be found from S_1, which involves 100 terms in its summation. The variance can be found from S_2, which involves $\binom{100}{2} = 4{,}950$ terms: not a calculation one would cheerfully undertake to do by hand, but something that is very easy to do with modern computers. With 100 targets, the third moment involves the summation of $\binom{100}{3} = 161{,}700$ terms. Although this is feasible with modern computers, some numerical analysis may be in order to ensure adequate precision in the final result. But the main point is that the expectation and variance can be found without solving the volley completely, and for applications that is a very handy feature.

The Helmbold Volley can also be generalized to allow for some types of collateral damage. For example, this could be done by setting

$$p_{ws}(\overline{z_t}) = \sum_{t' \in T} q_{ws}(t|t')v_{ws}(t'),$$

where $v_{ws}(t')$ is as before, but where $q_{ws}(t|t')$ is the probability that a shot from weapon w aimed at or allocated to target t' actually kills target t instead. Because we continue to assume that

$$\text{Prob}(\overline{z_1} \cup \overline{z_2} \cup \dots \cup \overline{z_n}) = \sum_{r=1}^{n} p_{ws}(\overline{z_r})$$

for all subsets of the target array, the resulting volley is still a point fire rather than an area fire volley.

6-3. The Burst Fire Volley. The Burst Fire Volley extends the Helmbold Volley to allow a burst of shots to be fired at an acquired target. It may be characterized by the following postulates.

BF-1: Each weapon acquires all of the T targets in the target array.

BF-2: Each weapon fires S_w bursts during the volley, allocated at the rate of one burst per target. The probability that weapon w allocates burst s to target t is $v_{ws}(t)$, and is independent of the allocations of other bursts from the same or any other weapon. The

probability that weapon w fires b rounds during a burst allocated to target t is $f_{wst}(b)$, and is independent of the other events that occur during the volley. All rounds fired in a burst are allocated to the same target as the burst.

BF-3: The probability that target t is killed if it is allocated b rounds in burst s from weapon w is $q_{ws}(t, b)$, and is zero otherwise, independent of the other events that occur during the volley.

It is clear that a Burst Fire Volley is equivalent to a Helmbold Volley in which

$$q_{ws}(t) = \sum_{b=0}^{\infty} q_{ws}(t,b) f_{wst}(b),$$

provided that the "shots" of the Helmbold Volley are identified with the bursts of the Burst Fire Volley. This justifies the location of the Burst Fire Volley entry in Table 3.

If the rounds in a burst are independently effective given the weapon-target combination involved, then $q_{ws}(t, b) = 1 - \{1 - q_{ws}(t, 1)\}^b$. In that case,

$$q_{ws}(t) = 1 - \sum_{b=0}^{\infty} \{1 - q_{ws}(t,1)\}^b f_{wst}(b).$$

6-4. The Hide-and-seek Volley. The Hide-and-seek Volley is a volley by a battery of W weapons against an array of T targets that satisfies the following postulates.

H-1: There are H hiding places and T targets, and $H \geq T$. Targets occupy hiding places uniformly at random and independently of each other, subject only to the condition that at most one target can occupy a given hiding place.

H-2: Each weapon fires exactly one shot, which it allocates to hiding place h with probability v_{wh}, independently of the actual location of the targets.

H-3: The probability that the shot from weapon w kills target t is $q_{wh}(t)$ when that shot is allocated to hiding place h and target t is occupying hiding place h, and is zero otherwise, independent of what other events occur during the volley.

Note that this is a volley by a battery of independently effective point fire weapons, and so it can be solved by finding the $p_w\left(\overline{z_t}\right)$ values. To do that, observe that the probability that weapon w kills target t is $v_{wh} q_{wh}(t)$ if target t occupies hiding place h, and since the probability that target t occupies hiding place h is $1/H$ for each $h = 1(1)H$,

$$p_w\left(\overline{z_t}\right) = H^{-1} \sum_{h=1}^{H} v_{wh} q_{wh}(t).$$

Then

$$P\left(z_{t_1} z_{t_2} \ldots z_{t_r}\right) = \prod_{w=1}^{W} \left\{1 - H^{-1} \sum_{n=1}^{r} \sum_{h=1}^{H} v_{wh} q_{wh}(t_n)\right\}$$

gives the basic event probabilities, and so provides the complete solution to the Hide-and-seek Volley.

The Hide-and-seek Volley is equivalent to a Helmbold Volley V^0 by a battery of W weapons against an array of T targets in which each weapon fires exactly one shot, allocated according to a uniform distribution over the T targets, and in which the kill probabilities are defined by

$$q_w^0(t) = \left(\frac{T}{H}\right) \sum_{h=1}^{H} v_{wh} q_{wh}(t).$$

This justifies the placement of the Hide-and-seek Volley entry in Table 3.

For the special case in which each weapon allocates its shot to a hiding place selected from the H hiding places according to a uniform distribution, $v_{wh} = H^{-1}$. Then

$$q_w^0(t) = \left(\frac{T}{H}\right) \overline{q_w}(t), \text{ where}$$

$$\overline{q_w}(t) = H^{-1} \sum_{h=1}^{H} q_{wh}(t).$$

If, in addition, $\overline{q_w}(t)$ is independent of t, then the Hide-and-seek Volley becomes equivalent to a DRR Volley by a battery of W weapons

against an array of T targets in which the kill probability is taken to be

$$q_w^0 = \left(\frac{T}{H}\right)\overline{q_w}.$$

6-5. The Karr Volley. Karr [1974] has analyzed in some detail a volley that he proposed as a model of the penetration of aircraft through a defended area, and for certain other types of penetration processes. We will extend Karr's results by providing formulas for the variance and correlation of the number of survivors. We paraphrase Karr's postulates for this volley as follows.

K-1: A battery of W weapons volleys against an array of T targets. The probability that weapon w acquires target t is $d_w(t)$ and is independent of other acquisitions made by the same or any other weapon.

K-2: A weapon that acquires one or more targets fires exactly one shot, which it allocates to a target chosen uniformly at random from among those it acquired, independently of the other events that occur during the volley.

K-3: The probability that target t is killed by weapon w is $q_w(t)$ if w allocates its shot to target t, and is zero otherwise, independent of the other events that occur during the volley.

Since a Karr Volley clearly is by independently effective point fire weapons, equation (21) applies. Helmbold [1992] shows that, when the targets are all alike, so that $d_w(t) = d_w$ for all $t = 1(1)T$, then

$$v_w(t) = T^{-1}\left\{1-\left(1-d_w\right)^T\right\}.$$

By virtue of the equivalence previously pointed out between the Karr and Helmbold Volleys, it follows that for the Karr Volley with exchangeably survivable targets, where A is any subarray of T_A targets,

$$P_{Ar} = \prod_{w=1}^{W}\left[1 - rq_w T^{-1}\left\{1-\left(1-d_w\right)^T\right\}\right]$$

for $r = 0(1)T_A$,

$$E\left(T_A^1\right) = T_A P_{A1},$$

$$\mathrm{Var}\left(T_A^1\right) = T_A\left(T_A - 1\right)P_{A2} + E\left(T_A^1\right)\left\{1 - E\left(T_A^1\right)\right\},$$

$$\rho_{jk} = \frac{P_{A2} - P_{A1}^2}{P_{A1} - P_{A1}^2} \quad \text{for } j \neq k, \quad \text{and}$$

$$P_{A[m]} = \binom{T_A}{m}\sum_{r=0}^{T_A-m}(-1)^r\binom{T_A-m}{r}P_{A(m+r)}$$

for $m = 0(1)T_A$.

Karr [1974] gives formulas for the distribution and expected number of survivors that are equivalent to (or special cases of) those given above, but does not provide formulas for the variance or correlation.

6-6. The Multishot Karr Volley. This volley generalizes the Karr Volley to allow each weapon to fire a number of shots, each allocated to a target chosen uniformly at random from among those it acquires. These results are new. In this volley, the weapons are independently effective, although they are area fire, rather than point fire weapons, as they are in the Karr Volley. Moreover, as will be apparent in the following development, the munitions are not independently effective point fire munitions. These observations justify the location of the Multishot Karr Volley in Table 3. Although this volley is not equivalent to a volley by independently effective point fire weapons, it seems appropriate to present and analyze it in the context of independently effective point fire weapons.

We treat only the case where the targets are all alike, and so write the acquisition probability as d_w and the kill probability as q_{ws}. When S_w is the number of shots fired by weapon w, the first order basic event probabilities can be found from

$$p_w(z_t) =$$
$$\left(1-d_w\right) + d_w\sum_{m=0}^{T-1}A_w(m,t)\prod_{s=1}^{S_w}\left\{1 - q_{ws}/(m+1)\right\},$$

where, as before, $A_w(m, t)$ is the probability

that weapon w acquires exactly additional m targets, given that it acquires target t. By Corollary 1.3, for the case at hand,

$$A_w(m,t) = \binom{T-1}{m} d_w^m (1-d_w)^{T-1-m_k}.$$

Helmbold [1992] shows that in general we will have

$$p_w\left(z_{j_1} z_{j_2} \cdots z_{j_r}\right) = \sum_{k=0}^{r} \binom{r}{k} d_w^k (1-d_w)^{r-k}$$

$$\sum_{m=0}^{T-r} A_w(m; j_1, j_2, \ldots, j_r) \prod_{s=1}^{S_w} \{1 - k q_{ws}/(m+k)\},$$

where

$$A_w(m; j_1, j_2, \ldots, j_r) = \binom{T-r}{m} d_w^m (1-d_w)^{T-r-m}$$

is the probability that exactly m targets other than the $j_1, j_2, \ldots, j_r$ are acquired. Observe that the expression given above for $p_w\left(z_{j_1} z_{j_2} \cdots z_{j_r}\right)$ shows that in the Multishot Karr Volley, the munitions generally are not independently effective point fire munitions (that is, neither equation (21) nor (25) applies).

Since the targets are exchangeably survivable, we may write the basic event probabilities for each of the single weapon volleys as

$$p_{wr} = p_w\left(z_{j_1} z_{j_2} \cdots z_{j_r}\right).$$

The basic event probabilities for a volley by the full weapons battery may then be obtained from

$$P_r = \prod_{w=1}^{W} p_{wr},$$

in accord with equation (18). This formula for P_r does not seem to reduce to any substantially simpler expression. However, it provides a complete solution to the Multishot Karr Volley and can be used in conjunction with equations (11) through (17) to compute numerical values for quantities of interest, such as the expectation and variance of the number of survivors.

7. VOLLEYS BY SYNERGISTICALLY EFFECTIVE WEAPONS

Volleys by synergistically effective weapons usually are more difficult to analyze than volleys by independently effective weapons, because no convenient general principles are available for expressing the effects of the whole battery of weapons in terms of smaller and more easily analyzed batteries. Of course, volleys by synergistically effective weapons against arrays of independently survivable targets often can be solved rather easily, as illustrated by the Artillery Volley. Even in that case, however, the effect of all the weapons in the entire battery had to be considered simultaneously. We now present two examples of volleys by synergistically effective weapons against targets that are not independently survivable.

7-1. The Bellwether Volley. This volley has been contrived to provide a solvable example of a volley by a battery of synergistically effective weapons against an array of targets that are neither independently nor exchangeably survivable. It is not put forward as having any important practical applications. It can be defined by the following postulates.

B-1: All weapons acquire all of the targets in the array.

B-2: One of the weapons is selected at random from among the battery of weapons to be the "bellwether" weapon. Let b_w be the probability that weapon w is chosen to be the bellwether weapon. The bellwether weapon then allocates its fire to a single target, which is chosen from the target array according to the probabilities $v_w(t)$ when weapon w is the bellwether weapon. All other weapons in the battery then allocate their fire to the same target as the bellwether weapon.

B-3: The probability that target t is killed during the volley is $q(t)$ if all weapons concentrate their fire against it, and is zero otherwise, independently of what other events occur during the volley.

The Bellwether Volley is easily ana-

lyzed. The probability that target t survives, given that weapon w is selected as the bellwether weapon, is

$$1-v_w(t)+v_w(t)\{1-q(t)\}=1-v_w(t)q(t).$$

This is true because target t survives if the bellwether weapon does not allocate fire to it, or if the bellwether weapon allocates fire to it, but it survives anyway. Because the bellwether weapon is selected at random, the probability that weapon w will be chosen as the bellwether weapon is $b_w = W^{-1}$, and so the first order basic event probabilities are

$$P(z_t)=\sum_{w=1}^{W} b_w\{1-v_w(t)q(t)\}=1-v(t)q(t),\text{ where}$$

$$v(t)=W^{-1}\sum_{w=1}^{W} v_w(t)$$

is the average probability that the battery will allocate all of its fire to target t. Now, the battery in a Bellwether Volley could be called a point fire *battery*, because it can kill at most one target per volley, and so the higher-order basic event probabilities are obtainable in terms of the first-order basic event probabilities, as follows.

$$\begin{aligned}P\left(z_{j_1}z_{j_2}\ldots z_{j_r}\right)&=1-P\left(\overline{z_{j_1}z_{j_2}\ldots z_{j_r}}\right)\\&=1-P\left(\overline{z_{j_1}}\cup\overline{z_{j_2}}\cup\ldots\cup\overline{z_{j_r}}\right)\\&=1-\sum_{n=1}^{r}P\left(\overline{z_{j_n}}\right),\end{aligned}$$

because

$$P\left(\overline{z_{j_1}}\,\overline{z_{j_2}}\right)=P\left(\overline{z_{j_1}}\,\overline{z_{j_2}}\,\overline{z_{j_3}}\right)=\ldots=0.$$

Therefore,

$$\begin{aligned}P\left(z_{j_1}z_{j_2}\ldots z_{j_r}\right)&=1-\sum_{n=1}^{r}v(j_n)q(j_n)\\&=1-\sum_{n=1}^{r}\left\{1-P\left(z_{j_n}\right)\right\}\\&=\sum_{n=1}^{r}P\left(z_{j_n}\right)-r+1.\end{aligned}$$

It can then be shown (see Helmbold [1992]) that

$$S_r=\sum_r P\left(z_j z_k\ldots z_r\right)=\binom{T-1}{r-1}S_1-(r-1)\binom{T}{r},$$

which can be written as

$$\begin{aligned}S_r&=\binom{T}{r}\{(r/T)S_1-(r-1)\}\\&=\binom{T}{r}\{1-r(1-S_1/T)\},\end{aligned}$$

where

$$S_1=\sum_{t=1}^{T}P(z_t)=T-\sum_{t=1}^{T}v(t)q(t).$$

Then we can write the generating function as

$$\begin{aligned}G(x)&=\sum_{r=0}^{T}(x-1)^r S_r\\&=(T-S_1)x^{T-1}+\{1-(T-S_1)\}x^T.\end{aligned}$$

Comparing this result with equation (3) shows that

$$P_{[T]}=1-(T-S_1)=1-\sum_{t=1}^{T}v(t)q(t),$$

$$P_{[T-1]}=T-S_1=\sum_{t=1}^{T}v(t)q(t),$$

and

$$P_{[m]}=0\quad\text{for } m=0(1)(T-2).$$

Also,

$$E\left(T^1\right)=T-\sum_{t=1}^{T}v(t)q(t)$$

and

$$\begin{aligned}\mathrm{Var}\left(T^1\right)&=2S_2+S_1-S_1^2\\&=(T-S_1)\{1-(T-S_1)\}.\end{aligned}$$

The Bellwether Volley is not a volley by independently effective weapons. For if it were, then its first order basic event probabilities would be

$$P(z_t) = \prod_{w=1}^{W} \{1 - v_w(t) q_w(t)\},$$

where $q_w(t)$ is the probability that target t would be killed if the fire of weapon w acting alone were allocated to it. But the required equality obviously does not hold in general. Nor would it hold even if it were assumed that

$$1 - q(t) = \prod_{w=1}^{W} \{1 - q_w(t)\},$$

that is, that the weapons are independently effective, conditional on the selection of target t as the one against which the entire battery's fire is concentrated. Moreover, in a Bellwether Volley

$$P(z_1 z_2) = P(z_1) + P(z_2) - 1,$$

which is not generally equal to $P(z_1)P(z_2)$. Hence, the Bellwether Volley is not a volley against independently survivable targets. Furthermore, it is clear that, in general, the Bellwether Volley is not a volley against exchangeably survivable targets, because there is no reason why the first order basic event probabilities $P(z_t) = 1 - v(t)q(t)$ should be independent of t. The observations of this paragraph justify the placement of the Bellwether Volley entry in Table 3.

7-2. A Redundantly Survivable Target Volley. By a redundantly survivable target we mean one that is able to survive several hits. More precisely, we assume that there is a redundancy number R that gives the maximum number of hits a target can tolerate without ill effect. That is, a target survives if it takes R or fewer hits during the volley and is killed otherwise (recall that in this paper we deal only with targets that are in one or the other of two possible states: dead or alive). With this notion of a redundantly survivable target in mind, we define the following version of a Redundantly Survivable Target Volley.

RST-1: Each weapon acquires all of the targets in the array.

RST-2: Each weapon fires exactly one shot, which it allocates to a target selected from the target array uniformly at random.

RST-3: The probability that the shot from weapon w hits target t is q if the shot is allocated to target t, and is zero otherwise, independently of what other events occur during the volley.

RST-4: Target t survives if, and only if, it takes no more than R hits in the course of the volley.

These postulates clearly describe a volley by a battery of synergistically effective weapons against an array of exchangeably survivable targets, confirming the placement of this volley in Table 3. It can be shown (see Helmbold [1992]) that

$$\begin{aligned} P_r &= P(z_1 z_2 \ldots z_r) \\ &= \text{Prob}\{(n_1 \le R) \cap (n_2 \le R) \cap \ldots \cap (n_r \le R)\} \quad (28) \\ &= \sum_{n_1=0}^{R} \sum_{n_2=0}^{R} \cdots \sum_{n_r=0}^{R} \frac{W!}{n_0! n_1! \ldots n_r!} q_{n_0}^{0} (q/T)^{W-n_0}, \end{aligned}$$

which provides the complete solution to this volley.

For the special case in which the targets survive if, and only if, they receive no hits, the redundancy number $R = 0$. In that case, the above expression for P_r reduces to

$$P_r = q_0^W = (1 - rq/T)^W,$$

which (as it should be) is identical to that for a DRR Volley in which the kill probability $q_w = q$ for all $w = 1(1)W$.

When the targets are singly redundant (that is, when $R = 1$), put

$$m = \sum_{j=1}^{r} n_j,$$

so that m ranges over the values $0(1)r$. Observe that, because in this case $n_j! = 1$ for each $j = 1(1)r$, equation (28) can be written as

$P_r =$

$$\sum_{n_1=0}^{1}\sum_{n_2=0}^{1}\cdots\sum_{n_r=0}^{1}\binom{W}{m}m!(1-rq/T)^{W-m}(q/T)^m$$

$$=\sum_{k=0}^{r}\binom{r}{k}\binom{W}{k}k!(1-rq/T)^{W-k}(q/T)^k,$$

where the last equality follows by observing that in the multiple summation exactly $\binom{r}{k}$ terms are such that

$$m=\sum_{j=1}^{r}n_j=k.$$

This is true because $\binom{r}{k}$ is the number of ways in which exactly k 1's can be assigned to the r n_j's (the other $r-k$ of the n_j's having the value of zero).

In general, when $R \geq 2$, the right side of equation (28) is not easily reduced to any substantially more compact expression. However, for the case $r = 1$ we note that

$$P_1=\sum_{n=0}^{R}\binom{W}{n}(q/T)^n(1-q/T)^{W-n},$$

which is sometimes useful, because the average number of survivors is given by $E(T^1) = TP_1$.

Observe that, by equations (13) and (28), when shots are allocated uniformly at random to the target array, we have

$$P_{[m]}(R)=\binom{T}{m}\sum_{r=m}^{T}(-1)^{m+r}\binom{T-m}{r-m}$$

$$\times\frac{W!}{(w-Rr)!(R!)^r}(q/T)^{Rr}(1-rq/T)^{W-Rr}$$

for the probability that exactly m of T targets each receive exactly R hits from a total of W shots. When $q = 1$, this is the same as the probability that exactly m cells each contain exactly R balls when a total of W balls are tossed randomly into T cells (compare this to Feller [1950]). Thus, the above generalizes the well-known occupancy problem of classical combinatorial probability theory.

8. CONCLUDING REMARKS

This concludes our presentation of the foundations of a general theory of volley fire models. In the course of it, we have reviewed previous work in this area and demonstrated that our approach not only powerfully unifies and extends previously used methods for solving volley fire problems, but often provides simpler and more intuitive solutions than have previously appeared. This general approach also shows that volley fire models generalize many of the classical probability problems in the theory of matchings, occupancy, and statistical mechanics. It also provides a useful system for classifying volleys into a few major categories to facilitate their solution by indicating the most appropriate solution method. In addition, it suggests potentially important new concepts, such as those for equivalent and complementary volleys. Moreover, it yields hitherto unpublished results.

In addition, various specific opportunities to extend or apply this treatment of the foundations of the general theory of volley fire were identified. Among them are the following.

1. Helmbold [1992] introduces the relation of complementarity between two volleys, and shows that, in general, the complements of volleys by independently effective weapons are volleys by synergistically effective weapons. How are those volleys by synergistically effective weapons characterized? What properties do they possess? What insights regarding the solution of volleys by synergistically effective weapons does this complementarity relationship afford?
2. How can the effects of successive volleys best be approximated? What error bounds apply to this approximation?
3. What limiting forms do volleys approach as various parameters (such as the number of targets, the number of weapons, and so forth) tend toward large or small values?

4. What optimization problems regarding volleys are most important, and what are their solutions?

Some larger issues which deserve attention in future research on volley fire models are as follows.

1. How can volleys against arrays of targets that may be in more than two states at the end of the volley be most efficiently analyzed?
2. What are the necessary and/or sufficient conditions under which explicit, closed-form solutions for the effect of successive volleys against arrays of targets be obtained?
3. What are the outcomes of volleys in which the target array is active, that is, returns fire? As far as we are aware, the deepest results on this have been reported by Gafarian and Manion [1989]. Versions in which the targets can countervolley have been treated by Helmbold [1966] (who, in a heuristic manner, derived the Lanchester square law equations from the limit of an alternating volley), Helmbold [1968] (although under rather restrictive assumptions), Bashyam [1970], and Zinger [1980].

Hopefully, calling attention to these challenging problems will stimulate analysts to devise original and imaginative solutions to them.

NOTES

Note 1. The probability of any complexion, and therefore of any event concerning the outcome of a volley, can be expressed in terms of sums and differences of basic event probabilities. For example,

$$\begin{aligned} P\left(z_j \overline{z_k} z_l \overline{z_m} z_n\right) &= E\left\{\tau_j(1-\tau_k)\tau_l(1-\tau_m)\tau_n\right\} \\ &= E\left\{\tau_j\tau_l\tau_n(1-\tau_k-\tau_m+\tau_k\tau_m)\right\} \\ &= P\left(z_j z_l z_n\right) - P\left(z_j z_l z_n z_k\right) \\ &\quad -P\left(z_j z_l z_n z_m\right) + P\left(z_j z_l z_n z_k z_m\right). \end{aligned}$$

This illustrates the following useful prescription given by Loeve [1960] for finding the probability of an event: (i) express the event as a sum of intersections of z's and $\bar{z}$'s (note that complexions are already in this form); (ii) replace the z's by τ's and the $\bar{z}$'s by $(1-\tau)$'s; (iii) express the result in the form of sums and differences of terms involving only products of the τ's; and then (iv) take the expectation. In taking the expectation, recall that the expectation of a sum is the sum of the expectations, and that

$$E\left(\prod_{n=1}^{N} \tau_{j_n}\right) = P\left(\prod_{n=1}^{N} z_{j_n}\right),$$

because the τ's are the indicators of the z's. Following this prescription produces for the probability of any event an expression involving only the sums and differences of basic event probabilities and justifies the assertion that, in principle, this is always possible. Of course, when there are many targets and the event involves complicated sums and differences of basic events, it may not be feasible to perform the calculations, even though explicit algorithms for them may exist.

Note 2. Let $H_A(x)$ be the generating function for the "tail" probabilities

$$\begin{aligned} P_{Am^+} &= \text{Prob(At least } m \text{ targets of subarray A survive the volley)} \\ &= \sum_{r=m}^{T_A} P_{A[r]} \\ &= P_{A[m]} + P_{A(m+1)^+}. \end{aligned}$$

Thus,

$$\begin{aligned} H_A(x) &= \sum_{m=0}^{T_A} x^m P_{Am^+} \\ &= G_A(x) + \sum_{m=1}^{T_A} x^{m-1} P_{Am^+} \\ &= G_A(x) + \frac{H_A(x)-1}{x}. \end{aligned}$$

Solving for $H_A(x)$ yields

$$H_A(x) = \frac{xG_A(x)-1}{x-1} = G_A(x) + \frac{G_A(x)-1}{x-1}.$$

Note 3. The n–th moment of the number of survivors can be expressed in terms of the first n basic sums, S_{Ak}, where $k = 1(1)n$. This can be shown as follows. Comparing the k-th derivative of $G_A(x)$, evaluated at $x = 1$, obtained from equation (3), with the same value, obtained from equation (4), shows that

$$G_A^{(k)}(1) = k!E\binom{T_A^1}{k} = k!S_{Ak}.$$

Since, by a standard result in combinatorial analysis (see section 24.1.4B of Abramowitz and Stegun [1964]),

$$\left(T_A^1\right)^n = \sum_{k=1}^{n} \Xi_2(n,k)k!\binom{T_A^1}{k},$$

where the coefficients $\Xi_2(n, k)$ are Stirling numbers of the second kind, it follows that

$$E\left\{\left(T_A^1\right)^n\right\} = \sum_{k=1}^{n} \Xi_2(n,k)k!S_{Ak}$$

$$= \sum_{k=1}^{n} \Xi_2(n,k)G_A^{(k)}(1).$$

These results express the n-th moment in terms of the first n basic sums, or in terms of the first n derivatives of the generating function. Abramowitz and Stegun [1964] tabulate the values of the Stirling numbers $\Xi_2(n, k)$ for $k = 1(1)n$ and $n = 1(1)25$.

Note 4. Suppose weights m_{Aj} and $\overline{m_{Aj}}$ are assigned to the events z_{Aj} and $\underline{z_{Aj}}$, respectively. Then the surviving weight of the targets in subarray A is

$$M_A^1 = \sum_{j=1}^{T_A} \left\{m_{Aj}\tau_{Aj} + \overline{m_{Aj}}\left(1-\tau_{Aj}\right)\right\}$$

$$= M_{A0} + \sum_{j=1}^{T_A} m_{Aj}\tau_{Aj},$$

where

$$M_{A0} = \sum_{j=1}^{T_A} \overline{m_{Aj}}$$

and

$$M_{Aj} = m_{Aj} - \overline{m_{Aj}} \quad \text{for } j = 1(1)T_A.$$

If hostile elements in the target array are intermingled with friendly units or other elements of value to the side controlling the battery of weapons, survival of the items of value might be assigned positive weights while survival of hostile elements are assigned negative ones.

Note 5. By postulates A-2 and A-3, the probability that target t survives all shots when it is located almost exactly at (x, y) is

$$P\left\{z_t \middle| (x,y)\right\} =$$

$$\int F_t \prod_{w=1}^{W} \prod_{s=1}^{S_w} \left\{\sigma_{ws}(u_{ws}, v_{ws})du_{ws}dv_{ws}\right\}.$$

Here F_t is the probability that target t survives, given the ground zeros of all the shots, as defined by postulate A-3, and $\int$ stands for the multiple integral

$$\int = \prod_{w=1}^{W} \prod_{s=1}^{S_w} \int_{(u_{ws}, v_{ws})}.$$

This integral may be written as

$$P\left\{z_t \middle| (x,y)\right\}$$

$$= \prod_{w=1}^{W}\prod_{s=1}^{S_w}\left[\int_{(u,v)} \left\{1 - D_{wst}(x-u, y-v)\right\}\sigma_{ws}(u,v)dudv\right]$$

$$= \prod_{w=1}^{W}\prod_{s=1}^{S_w}\left\{1 - \int_{(u,v)} \left\{1 - D_{wst}(x-u, y-v)\right\}\sigma_{ws}(u,v)dudv\right\},$$

where the first equality follows from the general theorem for converting multiple

integrals to products of single integrals, and the second follows from the assumption that σ is a probability density function.

Hence,

$$P(z_t) = \int_{(x,y)} P\{z_t|(x,y)\}\rho_t(x,y)dxdy$$

$$= \int_{(x,y)} \prod_{w=1}^{W}\prod_{s=1}^{S_w}\left\{1 - \int_{(u,v)} D_{wst}(x-u,y-v)\sigma_{ws}(u,v)dudv\right\}$$

$$\times \rho_t(x,y)dxdy,$$

and the formulas of the canonical volley against independently survivable targets apply to complete the solution for the basic event probabilities of an Artillery Volley.

Now, for the munitions to be independently effective, it is necessary that

$$P(z_t) = \prod_{w=1}^{W}\prod_{s=1}^{S_w} p_{ws}(z_t),$$

where

$$p_{ws}(z_t) =$$

$$\int_{(x,y)}\left\{1 - \int_{(u,v)} D_{wst}(x-u,y-v)\sigma_{ws}(u,v)dudv\right\}$$

$$\times \rho_t(x,y)dxdy$$

is the first order basic even probability for a volley that consists solely of shot s from weapon w. But, in general, $P(z_t)$ is not of the required form, and so the munitions generally are not independently effective. A similar argument shows that in general the weapons are not independently effective, either.

It is interesting to note that under some circumstances, the weapons can become independently effective in the limit. To show this, suppose that the damage function D_{wst} does not change with the shot number, so that it may be written as $D_{wt}(x-u, y-v)$. Suppose also that the number of shots S_w increases, while at the same time the distribution σ_{ws} "flattens out" in such a way that the product $S_w\sigma_{ws}(u, v)$ approaches the constant density N_w/A, where A has the physical dimensions of an area. Then, as Helmbold [1970] has shown, $P(z_t)$ approaches

$$P(z_t) = \exp\left(-\sum_{w=1}^{W} L_{wt}N_w/A\right)$$

$$= \prod_{w=1}^{W} p_w(z_t),$$

where

$$p_w(z_t) = \exp(-L_{wt}N_w/A)$$

is the first order basic event probability for a volley by weapon w acting alone against the target array. In the above equations,

$$L_{wt} = \int_{(u,v)} D_{wt}(u,v)dudv$$

is commonly called the lethal area for a munition of weapon w, when fired against target t.

Observe further that the targets are exchangeably survivable if L_{wt} is independent of t, in which case Corollary 1.3 applies to show that the expected fraction of targets that survive is

$$E(T^1/T) = P,$$

where

$$P = \exp\left(-\sum_{w=1}^{W} L_{wt}N_w/A\right)$$

does not depend on either t or w. This expression for the expected fraction of survivors is frequently used to obtain a quick estimate of the effect of fragmenting ordnance fired into a target area. We note that the variance of the fraction of survivors is

$$\text{Var}\left(T^1/T\right) = T^{-2}\text{Var}\left(T^1\right)$$

$$= T^{-1}P(1-P),$$

where the second equality follows from Corollary 1.3.

Note 6. Dixon [1953], Thomas [1956], and Helmbold [1966] have suggested various approximations for the effect of successive

DRR Volleys. Some of these are built around the idea of replacing the random variable T^1 with its expectation $E(T^1)$ and approximating the number of survivors after the n-th volley as

$$T^n \cong T^{n-1}\prod_{w=1}^{W}\left(1-q_w/T^{n-1}\right),$$

where T^n is the number of survivors after the n-th volley. This is called the expected value approximation. Although Helmbold [1966] presents some empirical support for this approximation, he provides no rigorous theoretical basis for it. Consequently, as noted by Karr [1974] and others, the conditions under which it is valid are not clear. In essence, the issue is one of establishing bounds on the error involved in making the expected value approximation. Although the present author has not pursued this matter, it may be that such bounds could be developed by using the formulas for the variance (or higher moments) of the number of survivors, together with Chebyshev's Inequality (or similar inequalities).

Another approach might be through the study of limiting forms. For example, we observe that the expected value approximation tends to be more accurate when the number of targets is large. This follows from the fact that when there are many targets, the argument given by Feller [1950, pp 69-74] applies to yield the following limiting Poisson approximations to the outcome of a DRR Volley:

$$E\left(T_A^1\right) \cong \mathrm{Var}\begin{pmatrix}T^1\\A\end{pmatrix} \cong \lambda_A,$$

$$\rho_{jk} \cong 0,$$

and

$$P_{A[m]} \cong \frac{\lambda_A^m}{m!}e^{-\lambda_A},$$

where

$$\lambda_A = T_A \exp\left(-\sum_{w=1}^{W} q_w/T\right).$$

In the limit, then,

$$P_{Ar} \cong \exp\left(-r\sum_{w=1}^{W} q_w/T\right) \cong \left(P_{A1}\right)^r,$$

which shows that the z's tend to become independent as the number of targets increases in which case the DRR Volley approaches a volley against independently survivable targets, for which the expected value approximation clearly is exact. For additional material related to the limiting forms of random allocations, see Kolchin, *et al.* [1978], and Choi [1987].

The study of limiting forms of volleys, and of approximations to the effect of several successive volleys, are two areas deserving additional research.

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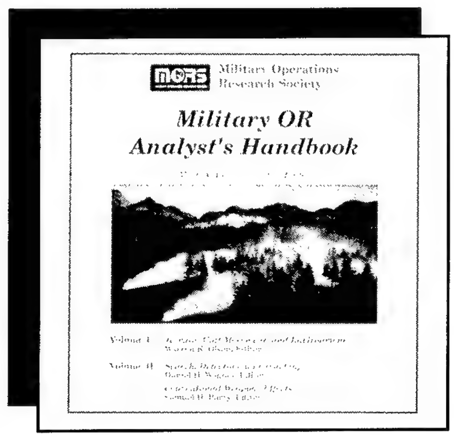

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Quantifying Counterforce and Active Defense in Countering Theater Ballistic Missiles

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ABSTRACT

This paper contains the formulation and analysis of a model to measure, compare, and contrast the effects of counterforce (pre-launch attack) and active defense (post-launch attack) against tactical ballistic missiles (TBM's). It is shown that without counterforce an active defense system could require an impractical number of weapons to counter incoming missiles and/or their warheads. This number is shown to decrease geometrically as effective counterforce is used, so that the expected number of warheads killed increases dramatically with counterforce that is only modestly effective. Actual distributions of warheads reaching the target area are shown to be complex mixtures of binomial distributions. It is shown that normal approximations to these distributions based on the easily-calculated means and variances often agree poorly with the actual distributions. This is especially true when using effective counterforce.

INTRODUCTION

In an earlier paper, (reference [1]) Conner, Ehlers and Marshall discuss the similarities between theater ballistic missile (TBM) defense and anti-submarine warfare (ASW). Both missions require searching, detecting, localizing, classifying, and finally attacking the object of interest. A great deal has been learned over the past fifty years on how to accomplish a successful ASW mission, and many of the lessons learned are pertinent to combating TBM's. Notice that ASW was never referred to as torpedo defense. Attempts were not made to kill the torpedo in the water; efforts were always concentrated on going after the launcher (the submarine) or the infrastructure necessary for it to operate. The reader interested in the ASW/TBM comparison is referred to that earlier paper. The purpose of this paper is to present and analyze a mathematical model of TBM launcher and missile flight operations so that comparisons can be made of the effectiveness of various strategies to counter the threat. The model presented here extends the earlier analysis and results found in reference [1].

Figure 1 shows a schematic of the operations of a TBM launcher and the missile assumed in this report. Launchers are expected to be stored in some fixed storage area. When hostilities are about to commence the launchers will move to a forward area for assembly, fueling and mating with the missiles. From there a launcher will move to its launch area, and after launch will return to the forward area to prepare for the next launch. We assume that each launcher has the potential to launch m missiles, after which it must be taken out of service for an extended time. The reason could be that it must undergo extensive repair and refit, or it could run out of missiles. We also assume that each

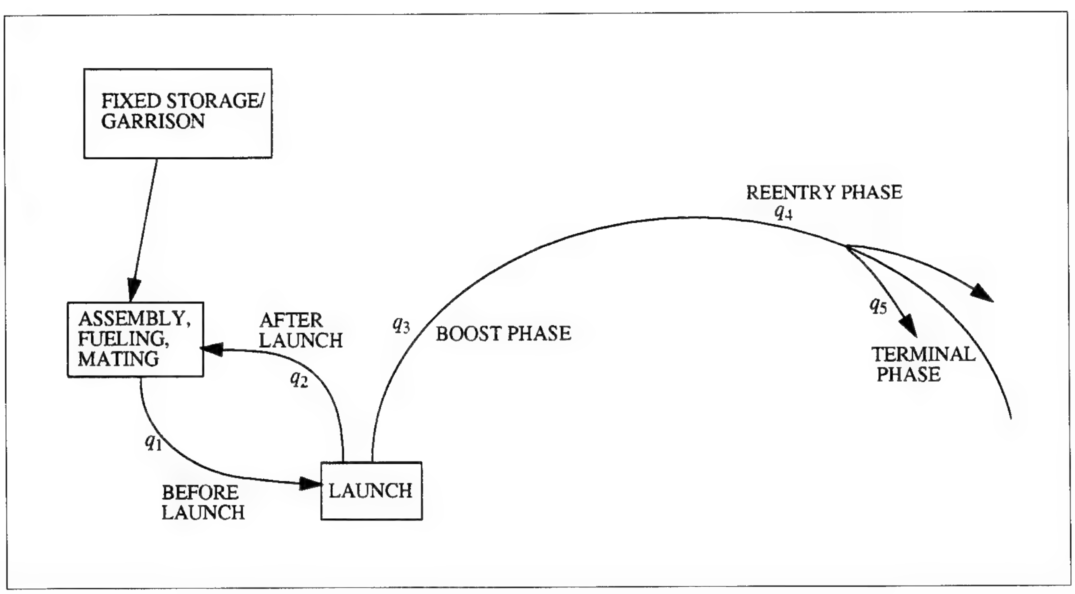

Figure 1: Schematic of Theater Ballistic Missile Operations

missile has n (≥ 1) warheads.

In this paper we assume that there are five phases in the TBM operation when the missile system could be attacked. These are

(a) Counterforce

1. Attack of the launcher with mated missile before launch between assembly area and launch site.
2. Attack of the launcher after missile launch either at the launch site or on return to assembly area.

(b) Active Defense

3. Attack of the missile during the boost phase,
4. Attack the missile on reentry before multiple warheads separate,
5. Attack each warhead in the terminal phase.

The effectiveness of attacking the system in each of these five phases is assumed to be summarized by a kill probability p_i for the i-th phase, or equivalently by a survival probability q_i, where $q_i = 1 - p_i$. Although it is more usual to formulate a model in terms of kill probabilities, survival probabilities are used in Section 2 because of the simplification that results in model development and presentation of results. Our objectives are to find the probability distribution, mean, and variance of the number of warheads reaching the target area from each launcher, and the expected number of weapons required in each phase, in terms of the maximum number of missiles per launcher (m), the number of warheads per missile (n) and the five survival probabilities for the five phases as shown in Figure 1. Using expressions for these quantities, in Section 3 we compare the effect of changing the model parameters to demonstrate that counterforce, with effectiveness measured by q_1 and q_2, will almost surely be a necessary part of a layered defense system; without at least a modest success rate in prosecuting the launchers, effective active defense may not be feasible.

1. THE ANTI-TBM MODEL

We build the mathematical model in stages following the missile's path from being mounted on the launcher to its or its launcher's destruction, or the arrival of its warheads in the target area. First we develop the probability distribution, mean and variance of the number of missiles that are successfully launched from a given launcher. These clearly will depend on the counterforce effort against the launcher. Next we derive the probability distribution, mean and variance of the number of missiles that survive the boost and reentry phases. Finally we find expressions for the probability distribution, mean and variance of the number of warheads that survive the final phase. The distribution of the warheads surviving to reach the target area is a complex mixture of Binomial probabilities. The section ends with numerical examples to illustrate the results. A detailed analysis using the model is presented in Section 2.

• Launcher Movement Phases

Let X be the number of missiles launched from a given launcher before it is either destroyed or has launched m missiles. We assume independent attacks each time the launcher attempts an outward journey to the launch site, and similarly for each time it attempts a return journey to reload. Thus X is a random variable that can take on any integer value from 0 (the launcher is destroyed on the first outward journey) to m (all attempts to destroy the launcher fail). Note that $X > i$ if and only if the launcher survives the first outward journey, and then survives i succeeding cycles back to the reload point and out again to the launch site. Thus

$$\begin{aligned} Pr\{X > 0\} &= q_1 \\ Pr\{X > 1\} &= q_1(q_1 q_2) \\ Pr\{X > 2\} &= q_1(q_1 q_2)^2 \\ &\cdots \\ Pr\{X > m-1\} &= q_1(q_1 q_2)^{m-1} \\ Pr\{X > m\} &= 0. \end{aligned}$$

The expected value of X is found by summing this cumulative tail distribution,

$$E[X]=\sum_{i=0}^{m-1} q_1(q_1q_2)^i=\frac{q_1\left(1-(q_1q_2)^m\right)}{\left(1-(q_1q_2)\right)}. \qquad (1)$$

This equation holds if both $0 \le q_1 < 1$ and $0 \le q_2 < 1$, and is equal to m when both q_1 and q_2 are equal to 1 (zero effect in killing the launcher before or after launch).

To find its variance we need to find its second moment. For a non-negative integer-valued random variable, say N, it is easy to show that

$$E\left[N^2\right]=2\sum_{i=1}^{\infty} i\,Pr\{N>i\}+E[N],$$

so

$$E\left[X^2\right]=$$

$$\frac{2q_1(q_1q_2)\left[1-m(q_1q_2)^{m-1}+(m-1)(q_1q_2)^m\right]}{\left(1-(q_1q_2)\right)^2}$$

$$+\frac{q_1\left(1-(q_1q_2)^m\right)}{\left(1-(q_1q_2)\right)}. \qquad (2)$$

This holds when both $0 \le q_1 < 1$ and $0 \le q_2 < 1$, When both q_1 and q_2 are equal to 1, $E[X^2] = m^2$. We find the variance of X in the usual way by subtracting the square of Equation (1) from Equation (2).

We now turn to finding the expected number of weapons required in the first two phases. Before attempting to do this it is necessary to make two important assumptions that are assumed to hold in all five phases. First, we assume that every time there is an opportunity to attack the launcher, the missile, or one of its warheads, this opportunity is taken and prosecuted with a single weapon. It may be that in practice more than one weapon is used, so that the numbers determined by the model in this report can be thought of as lower bounds. Second, the extreme case of some kill probability being zero in a given phase can be obtained in one of two ways, either (i) by not attempting an attack during that phase, or (ii) by attacking with a completely ineffective weapon system. In this paper we assume that the first of these is true; any time we use a p_i of zero (q_i of one) in phase i we assume no weapons are expended in phase i. The expected numbers of weapons required should not be interpreted as estimates of weapons requirements in actual operations. In this paper they are intended as an aid in gaining insight when comparing the effectiveness of changing kill probabilities in the various phases.

Let W_{BL} and W_{AL} be the numbers of weapons used in the "before launch" and "after launch" phases respectively against the launcher. Notice that if no missiles are launched, W_{AL} is zero (the launcher was destroyed on its first outward journey). It is easy to show that no matter how many missiles are launched from a given launcher, $W_{AL} = X$ and its first two moments are given by Equations (1) and (2).

By following the cycle of the launcher one can see that the cumulative tail distribution of W_{BL} is given by

$$Pr\{W_{BL}>i\} \begin{cases} =(q_1q_2)^i & \text{if } i=0,1,2,\ldots,(m-1), \\ =0 & \text{if } i \ge m. \end{cases}$$

Thus,

$$E[W_{BL}]=\frac{1-(q_1q_2)^m}{1-q_1q_2},$$

and by comparing this with Equation (1) we see that

$$E[W_{BL}] = E[X]/q_1. \qquad (3)$$

As our analysis progresses through the boost and reentry phases, expressions are found that require the probability mass function (pmf) of X. From the cumulative tail distribution above this is seen to be

$$p_X(0)=1-q_1,$$

$$p_X(i)=q_1(1-q_1q_2)(q_1q_2)^{i-1},$$

$$i=1,2,\ldots,m-1,$$

$$p_X(m)=q_1(q_1q_2)^{m-1}. \qquad (4)$$

• The Boost and Reentry Phases

The boost phase and reentry phase survival probabilities are q_3 and q_4 respectively (see Figure 1). Let the number of missiles surviving both of these phases (per launcher) be Y. Clearly this is also a random variable, and if we assume that the attempt to shoot down a given missile in either phase is independent of the outcomes of earlier or later attempts at other missiles, the conditional random variable $[Y \mid X]$ has a Binomial distribution with parameters X and q_3q_4. Thus $E[Y \mid X] = Xq_3q_4$ and $Var[Y \mid X] = Xq_3q_4\,(1 - q_3q_4)$. By unconditioning on X, the expected number of warheads surviving the reentry phase is

$$E[Y] = q_3q_4\,E[X] \tag{5}$$

where $E[X]$ is given by Equation (1). The variance of Y is found using the standard conditional variance argument,

$$Var[Y] = E_X[Var[Y \mid X]] + Var_X[E[Y \mid X]],$$

so

$$Var[Y] = q_3q_4\,(1 - q_3q_4)E[X] + (q_3q_4)^2 Var[X], \tag{6}$$

where Equations (1) and (2) are used to find $Var[X]$.

To find the pmf of Y, note that

$$p_{Y \mid X}(j \mid i) = b(j;\, i,\, q_3q_4)$$

where $0 \le j \le i$, $0 \le q_3q_4 \le 1$, and $b(j;\, i,\, p) = \binom{i}{j} p^j (1-p)^{i-j}$. Unconditioning on X we find

$$p_Y(j) = \sum_{i=j}^{m} b(i;\, j,\, q_3q_4)\,p_X(i),\ j = 0,1,2,\ldots,m, \tag{7}$$

where the $p_X(i)$'s are given in Equation (4).

Let W_B and W_R be the number of weapons used in the boost and reentry phases respectively against the missiles from a given launcher, and assume that exactly one weapon is used against each in each phase. If X survive launch, $W_B = X$ and W_R is a Binomial random variable with parameters X and q_3. Thus $E[W_B] = E[X]$, and $E[W_R] = q_3E[X]$.

• The Final Phase

In the final phase the probability that a given warhead survives an attack is q_5. Again we assume independence among all attempts to destroy incoming warheads. Let the number of warheads surviving the final phase from the i-th incoming missile be Z_i, $i = 1, 2, \ldots, Y$. Each Z_i is a Binomial random variable with parameters n and q_5, so $E[Z_i] = nq_5$ and $Var[Z_i] = nq_5(1 - q_5)$. Let the number of warheads surviving the final phase (per launcher) be H, so

$$H = \sum_{i=1}^{Y} Z_i\,.$$

Conditioning on Y, $E[H \mid Y] = nYq_5$ and $Var[H \mid Y] = YVar[Z_i] = nYq_5(1 - q_5)$. Unconditioning,

$$E[H] = nq_5E[Y] \tag{8}$$

and

$$Var[H] = nq_5(1 - q_5)E[Y] + n^2q_5^2 Var[Y], \tag{9}$$

where $E[Y]$ and $Var[Y]$ are given by Equations (5) and (6) respectively.

The pmf of H, $p_H(k)$, is found in a similar way by first conditioning on Y. If $Y = 0$ (no missiles survive through the reentry phase) no warheads can reach the target area, so $p_{H \mid Y}(0 \mid 0) = 1$. If $Y = j > 0$, H is the sum of j identically distributed binomials so that $p_{H \mid Y}(k \mid j) = b(k;\, nj,\, q_5)$. Unconditioning,

$$p_H(k) = \sum_{j=k}^{m} b(k;\, nj,\, q_5)\,p_Y(j),\ k = 0,1,2,\ldots,mn \tag{10}$$

where the $p_Y(j)$'s are given in Equation (7).

Let W_F be the number of weapons used in the final phase. If Y missiles survive the reentry phase and each carries n warheads, then $W_F = nY$. Thus the results on Y can be used to calculate the measures of interest on W_F.

Figure 2 demonstrates the model by showing the cumulative tail distribution of H for three different sets of survival probabilities. For all three cases the number of missiles per launcher (m) is 20, and the number of warheads

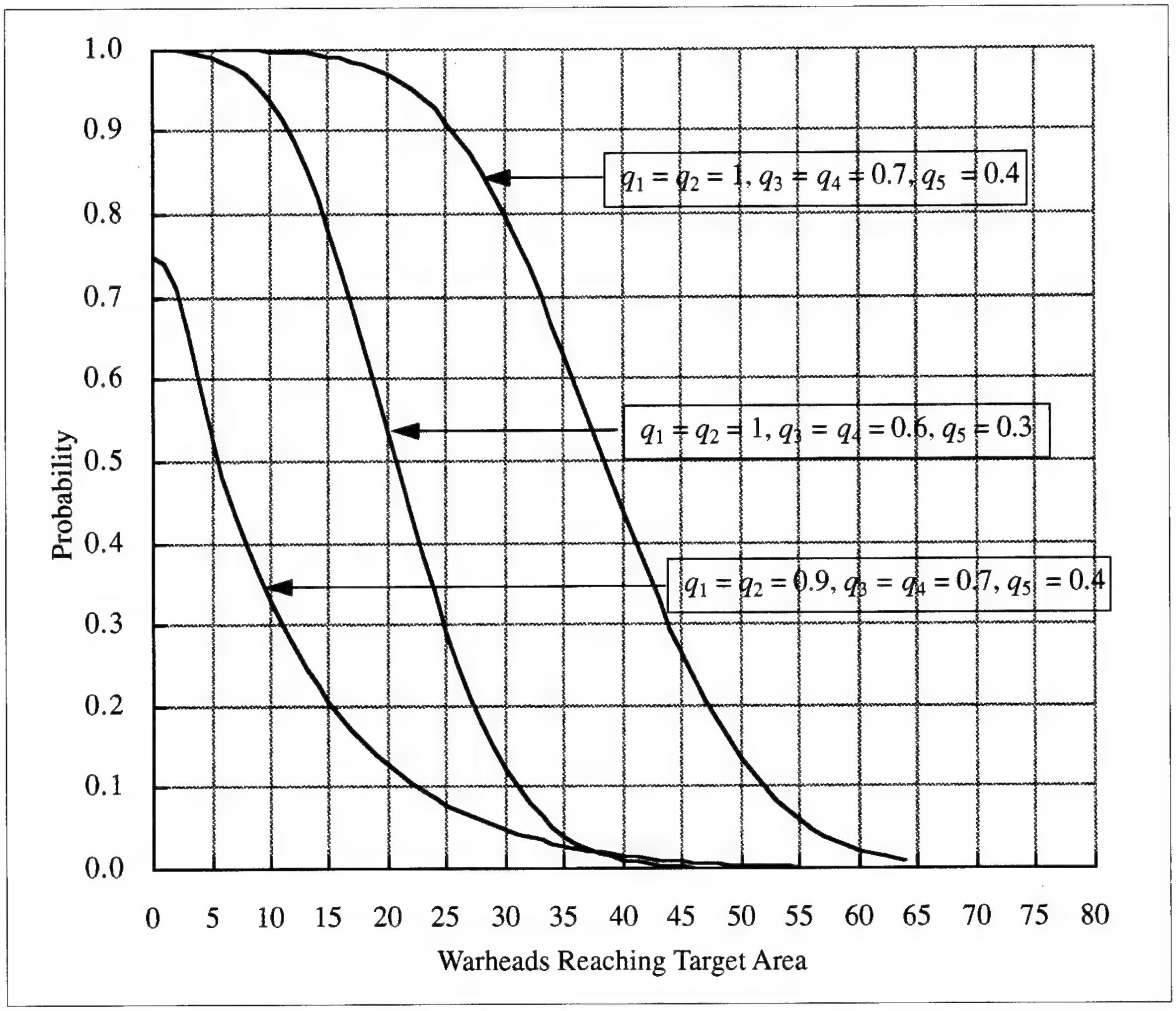

Figure 2: Cumulative Tail Distributions of Warheads Reaching Target Area

per missile (n) is 10. The rightmost curve is obtained using no (or completely ineffective) counter force ($q_1 = q_2 = 1$), boost and reentry survival probabilities (q_3 and q_4) of 0.7, and a final phase warhead survival probability (q_5) of 0.4. The center curve is obtained by decreasing q_3 and q_4 from 0.7 to 0.6, and q_5 from 0.4 to 0.3. The leftmost curve is obtained using the original set of parameters but decreasing both q_1 and q_2 from 1 to 0.9. Clearly a modest increase in kill probability in counterforce operations from 0 to 0.1 has a dramatic effect on the number of warheads reaching the target area. An increase in kill probability from 0 to 0.1 in the two phases of the launcher shows a drop in the 10-th percentile from 52 warheads to 23, compared to a drop from 53 to 31 for a similar increase in kill probability in the boost, reentry and final phases. Another way to interpret the three curves is to note that the chance of *at most* 20 warheads (10% of a potential of 200) reaching the target area is 3% for the base case. With a given improvement in active defense this increases to 46%, but if that improvement were made in counterforce instead of active defense it would increase to 87%. These numbers are shown in Column 2 of Table 1. Columns 3 through 6 show the expected number of weapons used in each phase. A small improvement in counterforce effectiveness sharply decreases the expected number of weapons required for active defense. Note that the zero entries in columns 3 and 4 result from the assumption that when $q_1 = q_2 = 1$, it is assumed that no counterforce is

Case	$Pr\{H \le 20\}$	$E\{W_{BL}\}$	$E[W_{AL}]$	$E[W_B]$	$E[W_R]$	$E[W_F]$
$q_1 = q_2 = 1, q_3 = q_4 = 0.7, q_5 = 0.4$	0.03	0	0	20	14	98
$q_1 = q_2 = 1, q_3 = q_4 = 0.6, q_5 = 0.3$	0.46	0	0	20	12	72
$q_1 = q_2 = 0.9, q_3 = q_4 = 0.7, q_5 = 0.4$	0.87	5.2	4.7	4.7	3.3	23

Table 1: Sample Output for Numerical Example

attempted.

The next section contains a more detailed analysis of the model as parameter values are varied.

2. MODEL ANALYSIS

Throughout this section results are demonstrated using kill probabilities p_1 through p_5 rather than survival probabilities, where $p_i = 1 - q_i$. We refer to a kill probability vector which is defined to be $(p_1, p_2, p_3, p_4, p_5)$. For example (0, 0.2, 0.3, 0.5, 0.6) represents no chance of killing the launcher in its outward journey to the launch site, a 20% chance of kill on its return journey to reload, a 30% chance of killing the missile in its boost phase, a 50% chance in its reentry phase, and a 60% chance of killing each warhead in the final phase.

Theater anti-missile defense today consists primarily of the use of the PATRIOT system in the final phase. The navy Aegis ship anti-missile defense system is currently being considered for modification for the reentry phase of the anti-TBM mission, and the army is developing the THAAD (theater high altitude air defense) system for this same phase. The air force is currently developing boost phase systems. Although some work has been done on detecting and destroying launchers prior to or after a launch, operational experience in Desert Storm showed that current systems and operational doctrine are ineffective. This current state can be modeled by setting p_1, p_2, p_3 and p_4 all equal to 0. We can set p_5 at some value depending on how well one believes the PATRIOT works. As a base case by which to measure possible system improvement we set p_5 to 0.7. Thus

$$\text{Base Case Kill Probability Vector} = (0, 0, 0, 0, 0.7). \tag{11}$$

Also as a base case we assume that a launcher can launch at most 20 missiles before requiring major overhaul, or before it runs out of missiles, so $m = 20$.

We look at three measures of effectiveness for the (random) number of warheads arriving in the target area, H. These are (i) the mean $E[H]$, (ii) the median, or that value h such that $Pr\{H \le h\} = 0.5$, and (iii) the ninetieth percentile, or that value h such that $Pr\{H \le h\} = 0.90$. We also look at the expected number of active defense weapons required ($E[W_B]$, $E[W_R]$, and $E[W_F]$), and the expected number of counterforce weapons ($E[W_C]$). We first look at today's case where there is only one warhead per missile ($n = 1$), and show how some performance measures are affected by improving kill probabilities in each of the first four phases. This is followed by a similar analysis when multiple warheads are considered.

• Single Warhead Analysis

The mean numbers of warheads (and hence missiles since we are assuming one warhead per missile) that arrive in the target area shown plotted in Figure 3 as a function of the kill probability at a particular stage. The figure contains three curves. All three start at the same point (0,6) because the expected number of warheads reaching the target area, $E[H]$, is 6 when $m = 20$, $n = 1$, the base case probabilities are given in (11), and Equations (1), (5), and (8) are used. We investigate the effect on $E[H]$ of increasing each of the four zero kill probabilities in (11) one at a time.

The upper curve is found by increasing the kill probability of either the boost (p_3) or reentry (p_4) phase from its base value of 0 up to 0.8. In either case it decreases linearly with a slope of -6. The middle and lower curves are obtained by increasing p_2 and p_1 respectively over the same range. The difference in the effect of a

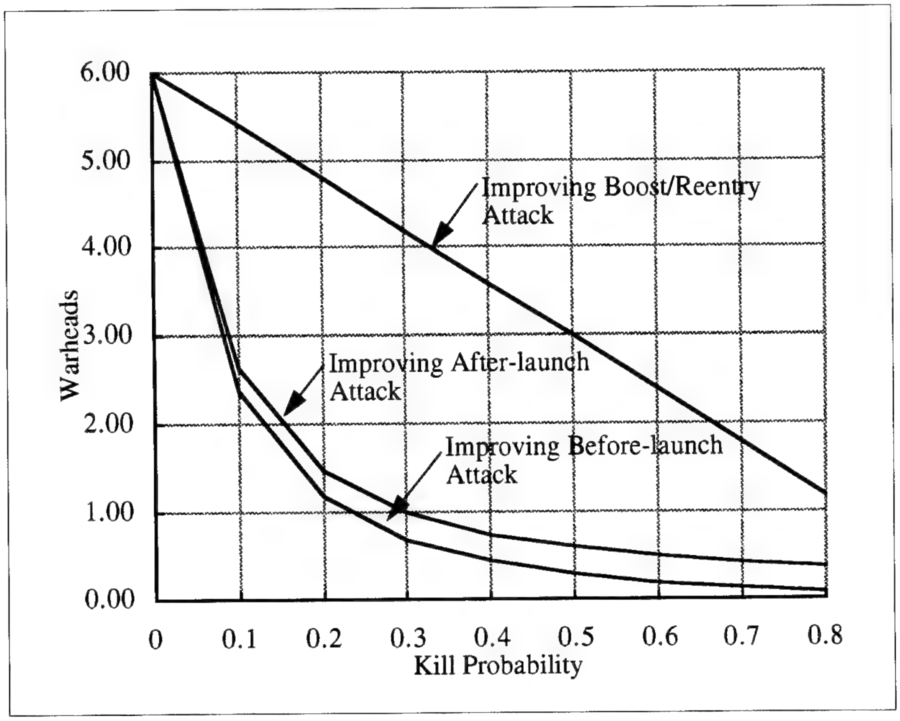

Figure 3: Mean Number of Warheads Reaching Target

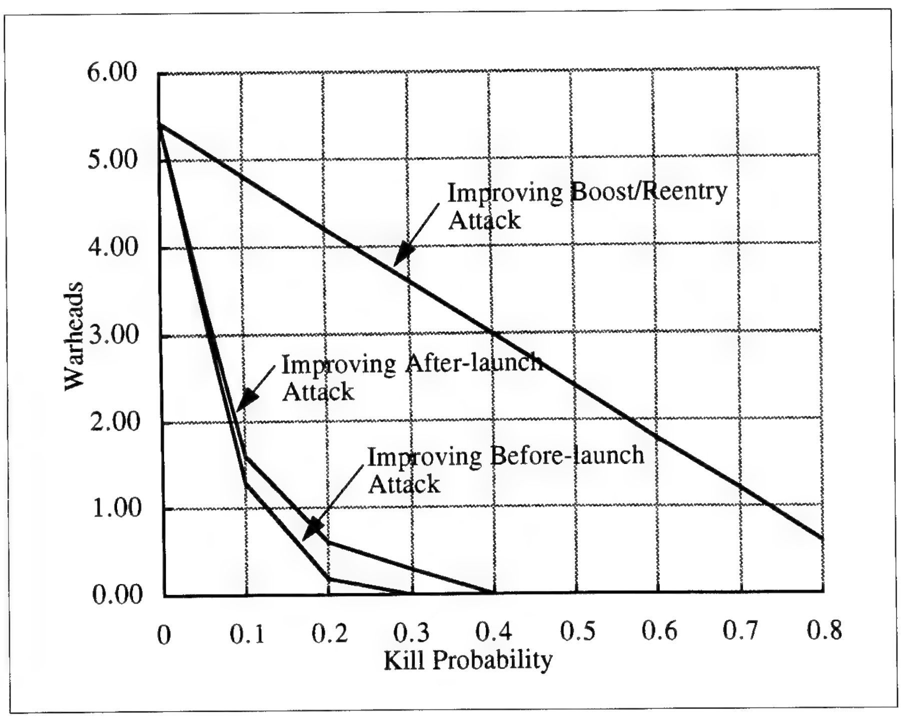

Figure 4: Median Number of Warheads Reaching Target

small increase in kill probability in the counterforce phases when compared to the active defense stages is dramatic; an increase from 0 to 0.1 in either the boost or reentry phases reduces $E[H]$ from 6 to 5.4, whereas this same increase in either of the counterforce stages reduces it from 6 to approximately 2.5. This significant improvement is caused by the fact that once a launcher (and its crew) is destroyed it can no longer fire missiles, causing a geometric reduction in $E[H]$. In the active defense stages a kill results in the destruction of only one missile. The small improvement in increasing p_1 rather than p_2 is caused by the fact that keeping p_1 at zero means the first missile from a launcher will be launched for certain, whereas increasing p_1 gives a chance to destroy the launcher before its first missile flies.

Figure 4 contains a similar analysis using the median number of warheads reaching the target area rather than the mean. Similar results are found. For the base case the median of H is 5.4. Increasing the boost or reentry kill probabilities from 0 to 0.1 reduces this to 4.8, whereas this increase in p_1 or p_2 reduces it to 1.3 and 1.6 respectively. In other words, using a kill probability vector (0.1, 0, 0, 0, 0.7) there is a fifty percent chance that fewer than 1.3 warheads will reach the target area, whereas using (0, 0, 0, 0.1, 0.7) or (0, 0, 0.1, 0, 0.7) this number is 4.8.

Figure 5 contains a similar analysis using the ninetieth percentile of the number of warheads reaching the target. For the base case there is a ninety percent chance that the number of warheads reaching the target area from a given launcher is no more than 8.2. Increasing the boost or reentry kill probabilities from 0 to 0.1 reduces this to 7.6 whereas an increase from 0 to 0.1 in p_1 or p_2 reduces it to 5.5 or 5.7 respectively. Although by using this measure of effectiveness there is less of a difference between improving counterforce and active defense, the difference is still significant.

We now turn to measuring the effects of changing kill probabilities on the expected numbers of weapons used in each phase. Starting from the base case we assume that a zero kill probability in a given phase indicates that no attempt is being made to kill the launcher or missile in that phase. Table 2 demonstrates typical results that can be obtained from the model. For the base case the expected number

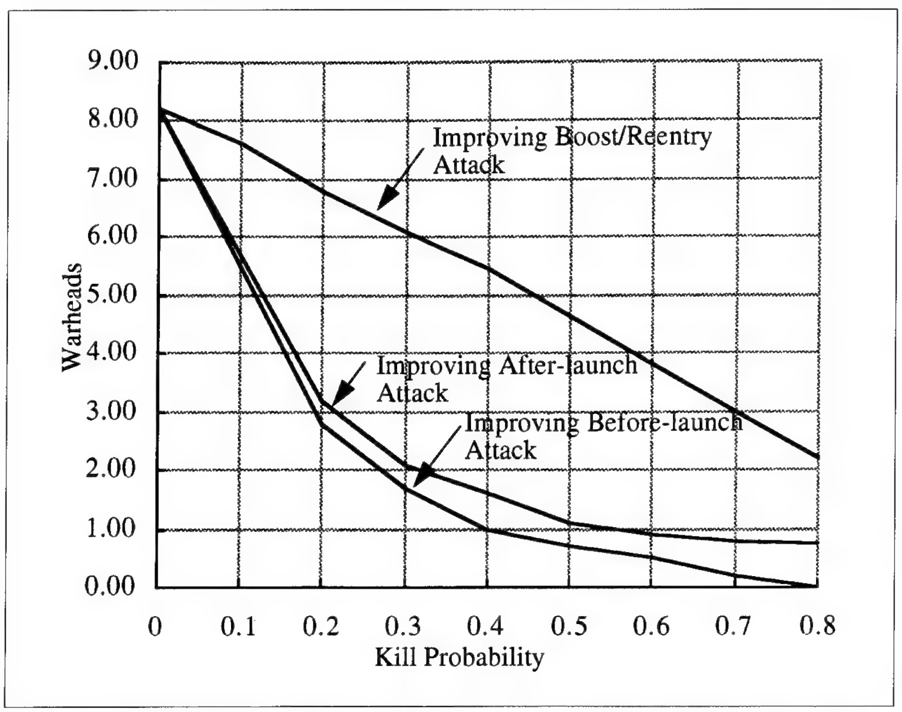

Figure 5: Ninetieth Percentile of the Number of Warheads Reaching Target

Kill Probability Vector	$E\{W_{BL}\}$	$E[W_{AL}]$	$E[W_B]$	$E[W_R]$	$E[W_F]$	Expected Warheads Killed/Weapon
(0, 0, 0, 0, 0.7)-Base Case	0	0	0	0	20	0.70
(0, 0, 0, 0.2, 0.7)	0	0	0	20	16	0.42
(0, 0, 0.2, 0, 0.7)	0	0	20	0	16	0.42
(0, 0.2, 0, 0, 0.7)	0	4.94	0	0	4.94	1.88
(0.2, 0, 0, 0, 0.7)	4.94	0	0	0	3.95	2.11

Table 2: Effect of Increasing Kill Probabilities on Weapons Numbers and Effectiveness

of weapons used per launcher when no attempt is made to destroy the missile before the final phase, and assuming one weapon for each warhead, is equal to the number of missiles times warheads per missile that a launcher can launch. In this example that is 20. Also for the base case the expected number of warhead kills per weapon is equal to the final phase kill probability as should be expected. The remaining rows in Table 2 show the effect of increase the kill probability of each phase in turn from 0 to 0.2. Note the dramatic drop in the requirement for weapons in the final phase by having a modest effectiveness in counterforce versus the same effectiveness in the boost or reentry phases. In those phases a modest kill probability significantly increases the warhead kills/weapons used ratio.

Multiple Warhead Analysis

We repeat the analysis of the last section using the same base case kill probability vector shown in (11) and twenty missiles per launcher ($m = 20$), but in this section we assume each missile carries ten warheads ($n = 10$). The same

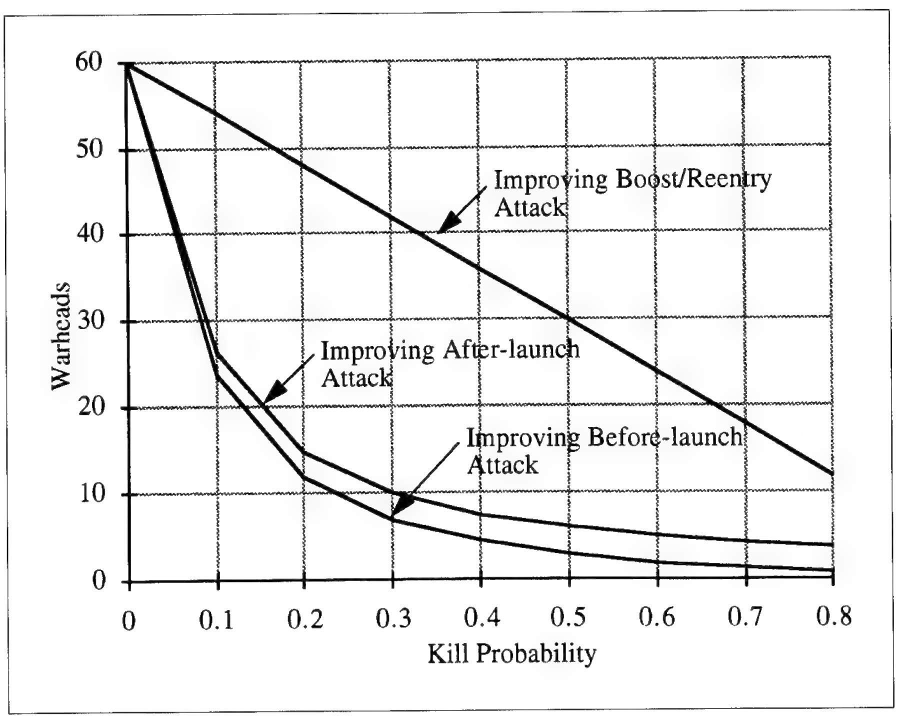

Figure 6: Mean Number of Warheads Reaching Target, Ten Warheads per Missile

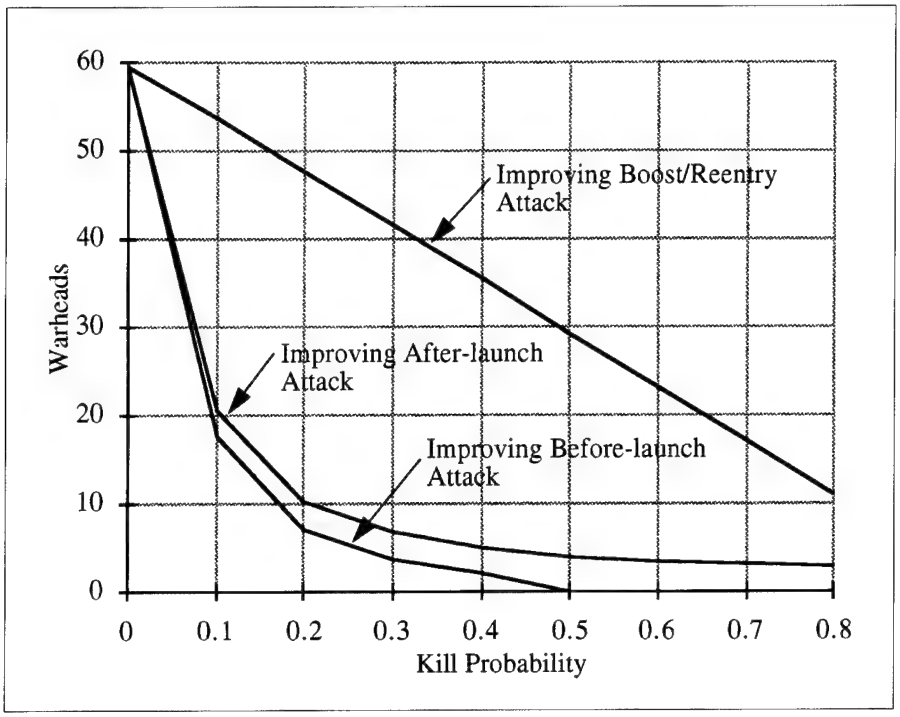

Figure 7: Median Number of Warheads Reaching Target, Ten Warheads per Missile

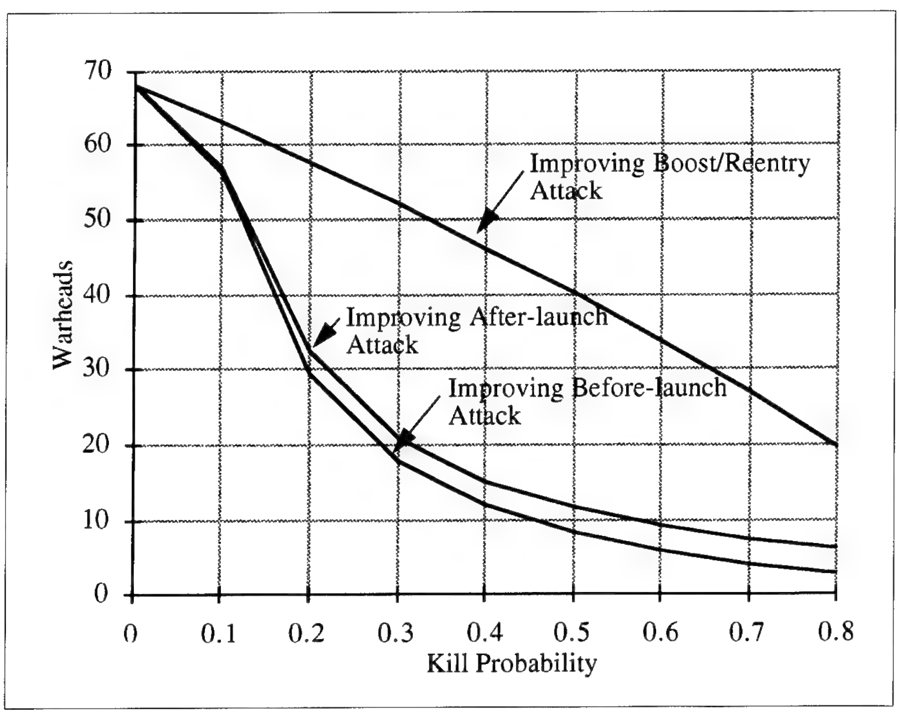

Figure 8: Ninetieth Percentile of Warheads Reaching Target, Ten Warheads per Missile

types of results are illustrated in Figures 6, 7 and 8 as were seen in Figures 3, 4 and 5. In fact since the mean is linear in n the curves in Figure 5 are the same as those in Figure 3 except the vertical scale has changed by a factor of 10. There is no simple relationship between the median or the ninetieth percentile and n, although over some of the range of the kill probability the relationship appears to be approximately linear. For example, from Figure 4 with $n = 1$ we see that a median number 2 for H (90% kill of the twenty possible warheads) can be achieved if p_1 or p_2 are close to 0.08, whereas in the boost or reentry phases we would need p_3 or p_4 to be 0.56 to achieve this success. From Figure 5 with $n = 10$ we see that a median number 20 for H (90% kill of the two hundred possible warheads) can be achieved if p_1 or p_2 are close to 0.1, whereas in the boost or reentry phases we would need p_3 or p_4 to be 0.65. Similarly, from Figure 5 we see that to achieve a ninetieth percentile of 2 when $n = 1$ requires either a p_1 or p_2 of about 0.28 or a p_3 or p_4 of 0.81; from Figure 5 a ninetieth percentile of 20 when $n = 10$ requires either a p_1 or p_2 of about 0.30 or a p_3 or p_4 of 0.79.

Table 3 shows the expected numbers of weapons required at each stage and the expected warhead kills per weapon when $n = 10$. By comparing the results with those in Table 2 it is clear that the required expected numbers of weapons at the counterforce, boost, or reentry phases do not change when warheads per missile increase from 1 to 10, but the number of weapons in the final stage increases by a factor of ten. These results should be expected since a successful kill at any phase before the warheads separate is assumed to kill all n warheads. Note that, as n increases, the expected number of warheads killed per weapon increases significantly, the earlier one can attack the TBM operation. In other words, counterforce is increas-

Kill Probability Vector	$E[W_{BL}]$	$E[W_{AL}]$	$E[W_B]$	$E[W_R]$	$E[W_F]$	Expected Warheads Killed/Weapon
(0, 0, 0, 0, 0.7)-Base Case	0	0	0	0	200	0.70
(0, 0, 0, 0.2, 0.7)	0	0	0	20	160	0.84
(0, 0, 0.2, 0, 0.7)	0	0	20	0	160	0.84
(0, 0.2, 0, 0, 0.7)	0	4.94	0	0	49.4	3.41
(0.2, 0, 0, 0, 0.7)	4.94	0	0	0	39.5	4.23

Table 3: Expected Weapons Numbers and Effectiveness with Ten Warheads per Missile

Kill Probability Vector	Ten Warheads per Missile ($n = 10$)		One Warhead per Missile ($n = 1$)	
	Median	Normal Approximation	Median	Normal Approximation
(0, 0, 0, 0, 0.7)-Base Case	59.3	60.0	5.4	6.0
(0, 0, 0, 0.2, 0.7) or (0, 0, 0.2, 0, 0.7)	47.6	48.0	4.2	4.8
(0, 0.2, 0, 0, 0.7)	10.3	14.8	0.6	1.5
(0.2, 0, 0, 0, 0.7)	7.1	11.9	0.2	1.2
(0.2, 0.2, 0.3, 0.5, 0.7)	NA	2.3	NA	0.2

Table 4: Normal Approximation for the Median

ingly effective as the number of warheads carried by the missile increases.

Normal Approximations

For given values of m, n, and a kill probability vector, it is easy to calculate the expected value of H using Equations (1), (5), and (8); likewise one can easily find the variance using Equations (1), (2), (5), (6), and (9). But to find percentiles such as the median or the ninetieth percentile requires the distribution function of H, a much more complex calculation using Equations (4), (7), and (10). These equations were used to find the curves in Figures 2, 4, 5, 7, and 8. Recall that H is not a simple sum of independent random variables, but results from a complex set of five random events, the first two of which have a truncated geometric distribution, the next two a conditional binomial distribution, and the last is a random sum of these weighted binomials. Even so, one might suspect that its distribution is approximately normal for at least some range of the parameter values, in which case the percentiles can be estimated using only the mean and variance of H. We investigate the appropriateness of a normal approximation for the median and ninetieth percentiles of H in this section.

Since the normal is a symmetric distribution its mean and median are equal. Table 4 contains actual medians and normal approximations for the base case and kill probability vectors, and an example that assumes positive kill probabilities in all five stages. The normal approximation seems to perform reasonably well for the ten warhead case when there are zero kill probabilities in the counterforce stages; it does less well in the single warhead case. When p_1 and/or p_2 are significantly larger than

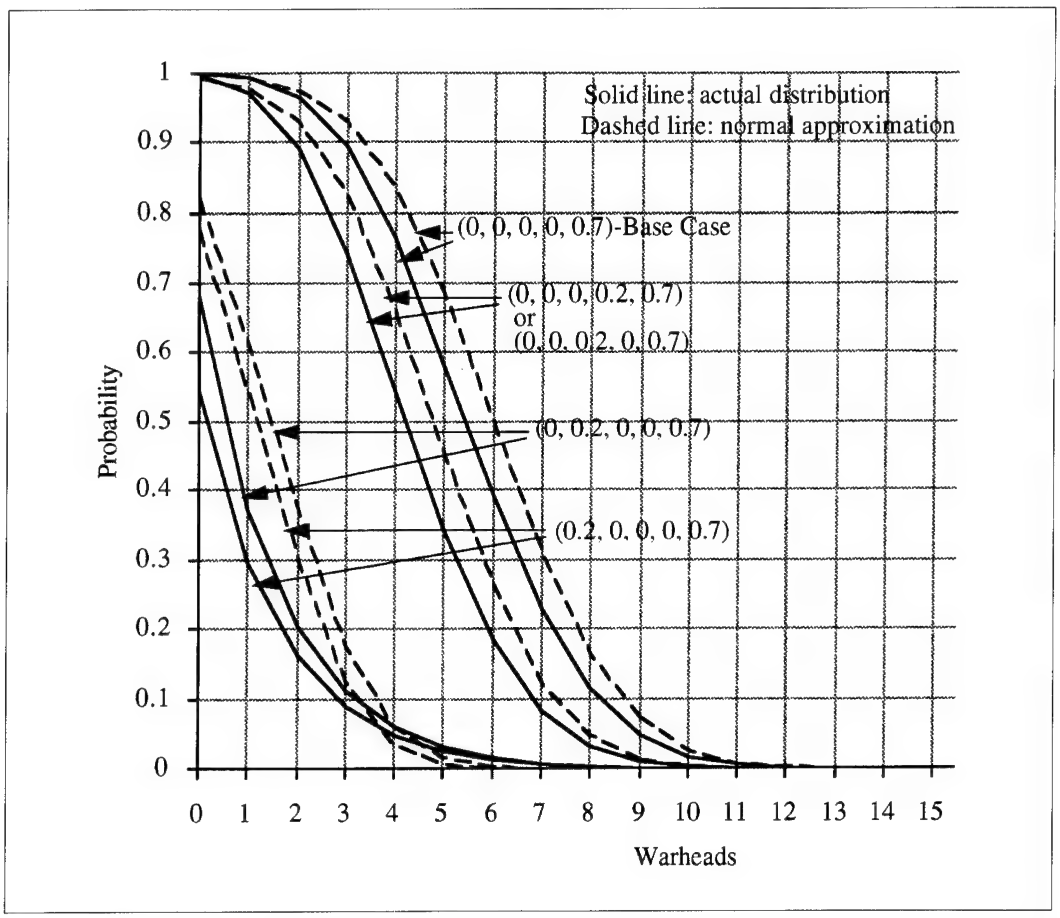

Figure 9: Cumulative Tail Distributions and Normal Approximations, n = 1

zero, the distribution of H is highly skewed and the normal approximation for the median is poor. The entries NA (not applicable) in the table indicate that the probability that H is zero is larger than 0.5 so that no median value exists.

Figure 9 contains cumulative tail distributions (solid lines) and normal approximations (dashed lines) for the kill probability vectors in Table 3 and one warhead per missile. For none of the examples is the normal approximation close to the actual distribution except in the extreme tails. It is particularly poor when there is a positive probability of kill by counterforce.

Figure 10 contains cumulative tail distributions (solid lines) and normal approximations (dashed lines) for the kill probability vectors in Table 3 and ten warheads per missile. When there is no counterforce the normal approximation is close to the actual distribution over the whole range, but again there are significant differences when there is a positive probability of kill by counterforce.

As one might expect the approximation does quite well when H is a fixed (non-random) sum of binomial random variables. Since this number is considerably larger when multiple warheads are present it does significantly better in this case. With positive counterforce probabilities the truncated geometric distribution of the number of missiles launched leads to skewing of the distribution of H. In this case the normal approximation shows significant error.

It is not recommended that the normal approximation be used for the median (or equivalently that the median and mean be assumed to take on the same value). Nor is it recommended that it be used as an approximation to the tail distribution unless multiple warheads are assumed to be present and the

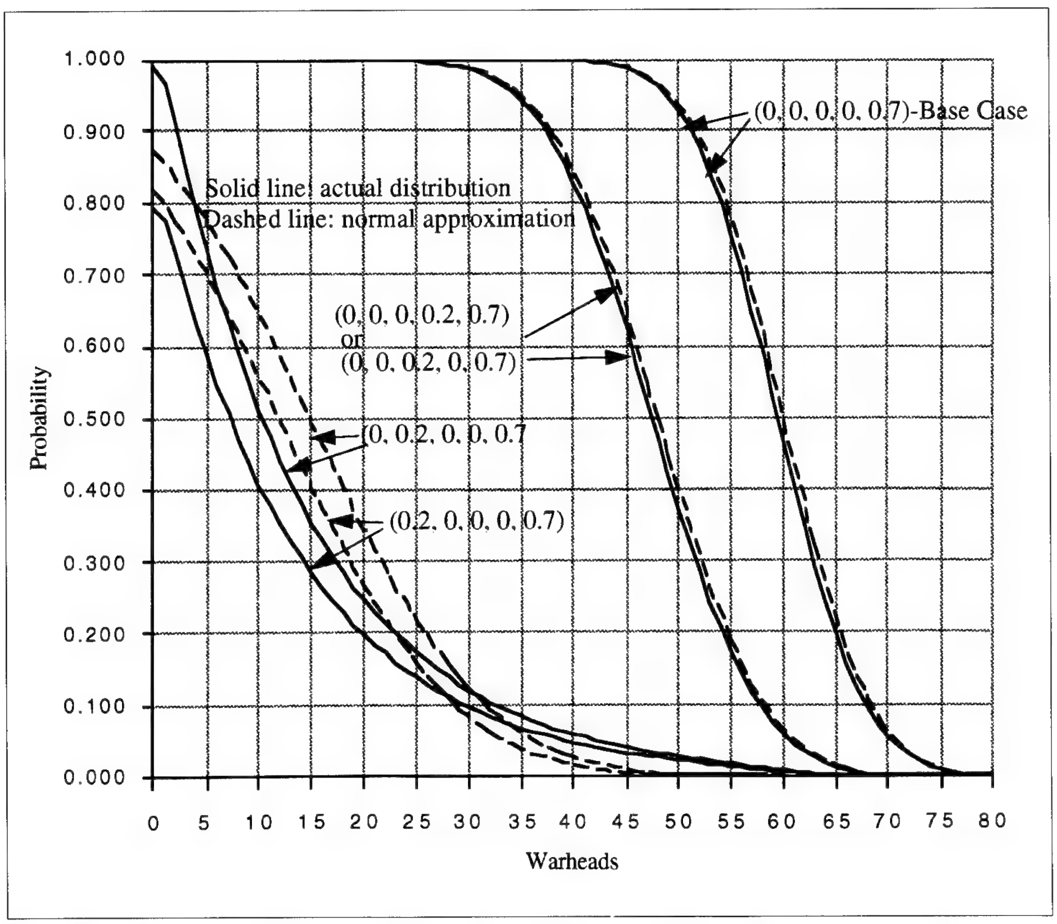

Figure 10: Cumulative Tail Distributions and Normal Approximations, $n = 10$

only significant source of uncertainty is in the final stages of the TBM operation.

3. CONCLUSIONS

The model in this report shows that both counterforce and active defense will form essential parts of any future successful system for theater ballistic missile defense. Without counterforce it will be relatively easy for the enemy to overwhelm a feasible active defense system. A system that can successfully destroy launchers and their crews will provide considerable leverage in reducing the numbers of active defense weapons required; this leverage increases dramatically as the number of warheads on each missile increases. The model allows the calculation of percentiles of the numbers of warheads destroyed rather than simple expected values.

Past experience in finding and destroying launchers has demonstrated little success. As was discussed in Conner, Ehlers, and Marshall [1993], success will most likely require a far more structured approach than has been used. A model for such a structure is that used in antisubmarine warfare where great experience has been gained in the past fifty years at finding and destroying torpedo underwater missile launchers. It is expected that the successful counterforce against launchers on land will require efforts in cueing, search, detection, localization, classification and destruction. Current efforts can be thought of as attempting to skip from cueing (for example, flaming datum information after launch) to attack. Future reports will consider how one might best accomplish the in-between phases to produce successful counterforce against mobile missile launchers.

REFERENCES

Conner, George W., Ehlers, Mark A., and Marshall, Kneale T., "Countering Short Range Ballistic Missiles," Technical Report NPS-OR-93-009, Department of Operations Research, Naval Postgraduate School Monterey, CA 93940, January, 1993.

FOOTNOTES

[1] It is understood that a launcher may employ a number of tactics on its way to or from the launch site, such as stopping in hide sites. The model summarizes the effects of these strategies in a single survival or kill probability.

Heritage Department

E.P. Visco, FS

Office of the Deputy Under Secretary of the Army (Operations Research)

From time to time the Editors of MOR will bring to these pages papers which have been hidden from view for a long time. The papers will be chosen to illustrate early applications of military operational analysis. Purposes for re-printing older work include keeping the heritage of our practice alive and contributing to the education of younger analysts. The Editors welcome your comments about this department, positive and negative, and also welcome suggestions for candidates for these pages.

It is fitting that we initiate the series with a paper of F. W. Lanchester, generally seen as a "father figure" by military operations analysts. More well known chapters of Lanchester's 1916 book covering the differential exchange or attrition equations are readily available and have been the source of endless discussion and writings. I have chosen lesser known parts of Lanchester's work because I believe that the work does fall in the category of military operational analysis and that the work is highly unlikely to have been read by many of our audience—at least, not in recent days.

The second paper in this series is a report by Mr. E.D.T. Strong, one of the early British practitioners of Operational Research. This is a report of the trials and tribulations of doing operations research in the theater, complete with a number of amusing anecdotes. This was presented as a banquet speech at the 10th International Symposium on Military Operational Research at the Royal Military College of Science, Shrivenham, UK 1993.

Frederick W. Lanchester's famous book, ***Aircraft in Warfare. The Dawn of the Fourth Arm,*** *was published in 1916 by Constable and Company, London. The book is a compilation of papers previously published in a journal,* ***Engineering,*** *in the words of the author "... covering a period from September to December, 1914. The text and order of the original articles have been preserved in the present volume, and thus the matter appears under the dates of its original publication. Revision has, in the main, been confined to ordinary legitimate corrections, the articles having been regarded and treated to all intents and purposes as a first proof: The last two chapters, however, include new matter; they are for this reason undated." For the interested reader, the last two chapters are: Chapter XVIII (Retrospect—The Scope and limitations of the Work. Supplementary Notes on the N-Square Law Air Raids and the Value of Numbers. A Further Note on Aircraft and Submarine. The Strategic Employment of Aircraft on a Large Scale) and Chapter XIX (Air Raids—Some Questions of National Defence. Power of Aggression as Affected by Radius of Action. Air Raids as Affecting the Naval Outlook. Aeronautical and Naval Defence Indissolubly Associated. Future of Air Power: Essentially a National Question. Categorical Statement of Recommendations for Future Policy.) Are you intrigued?*

I have chosen to start these proceedings with the third chapter. Each paper will be presented here in the format as originally published to the extent possible, commensurate with our modern printing and editing processes. In the case of Lanchester's book section numbers start with the first chapter and continue sequentially, without renumbering within chapters; thus, Chapter III starts with section 8. English spelling is used throughout.

Aircraft in Warfare
CHAPTER III
(September 11th, 1914)

Strategic and Tactical Uses of the Aeronautical Arm. Aircraft as Vulnerable to Gun-Fire. Armour and Altitude as Means of Defence.

§8. *Strategic and Tactical Uses of the Aeronautical Arm.* In the present distribution of the cavalry Arm, the distinction between the strategic and tactical uses of cavalry is clearly recognised. For purely tactical purposes it is customary to attach one or more squadrons, usually a regiment of cavalry, to each infantry division. The main cavalry force on the other hand,—known as the independent cavalry,—constitutes a separate command, taking general instructions from the headquarters staff. The independent cavalry may be engaged in operations of strategic import, as in the conduct of a reconnaissance in force, or in the execution of a wide turning or outflanking movement, or in the countering of such a movement on the part of the enemy. Alternatively it may be employed in its tactical capacity, its full weight being thrown at some critical moment into the fighting line, it may be to attack and destroy the cavalry of the enemy, to raid and capture or put out of action his artillery, to harass him in retreat, or to convert a retreat into a rout. The *divisional* cavalry are, generally speak-

ing, employed for the latter—tactical—duties only.

In a similar manner aircraft are capable of employment in duties of both strategic and tactical import, and accordingly will probably need to be divided into divisional and independent commands. Thus there is the machinery of strategic reconnaissance, whose function it is to inform the headquarters staff of the main disposition and movements of the enemy's forces, positions of his depôts, magazines, etc., points of concentration and strength of his reserves, and last, but not least, his main and perhaps auxiliary lines of communication. On the tactical side there are similarly many duties to be carried out, analogous to those at present performed by cavalry; there are also duties which must be regarded as new, brought into being by the peculiar power and capacity of the aeronautical Arm; these are, in the main, such as would indicate control by the divisional command.

§9. *The Strategic Scout and its Duties.* The strategic value of the aeroplane depends mainly upon its utility for the purpose of reconnaissance; briefly it is its value as an informer, rather than as a fighter, that is of service to the headquarters staff. The duties of a machine thus acting are necessarily of an entirely different character from those of a machine employed in the minor operations of the field, whether for tactical scouting, direction of gun-fire, or otherwise. Firstly, the flight range or radius, as determined by petrol capacity, is a far more important factor in its design, since it will require to operate over a large area, and to cover long distances over the enemy's territory, where any renewal of fuel supply is impossible; secondly, its flight speed must be such as to render it reasonably secure against pursuit. Anything serious in the direction of armour or armament will be entirely out of place, since under no circumstances will such a machine be required to act in a combative capacity; its defence lies in its speed. It appears from all reports that the duties in question are such as to require an observer (probably a staff officer) of mature knowledge from a military standpoint, with considerable flying experience, possessing something of an intuition for reading the meaning of the incomplete and fragmentary indications which are obtainable from high altitude observation. It is evidently not impossible for a strategic scout (as we may term the machine under discussion) to descend to low altitude in pursuit of more accurate and precise information; but it is always to be remembered that any such manaeuvre is dangerous to an unarmoured machine; it may be too easily shot down or destroyed by shrapnel. In this latter event it must be regarded as having failed in its purpose. The possession of a wireless installation may be assumed, but, in the event of the machine being lost, the fact that reports had already been transmitted to headquarters would in no way mean that the machine had completely fulfilled its mission.

The work done by the strategic scout thus comprises the gleaning of information hitherto only to be obtained by espionage or by a reconnaissance in force—that is to say, by a large force of cavalry with supports of horse artillery and infantry, often involving considerable fighting and loss. It is quite improbable that aeroplane scouting will prove an entire substitute for such reconnaissance; it may be said that cavalry can *feel* and *act* where the air-scout can only *see* and *report*, but, as a prelude to cavalry reconnaissance, and as an auxiliary thereto, the services of the strategic scout should prove of the utmost utility. It will, at least, enable the cavalry force acting at a distance from its base, frequently in the rear of the enemy, to keep in constant touch with headquarters, and thus relieve the despatch rider of one of his most difficult and dangerous tasks. In service of this character it would seem probable that a *flight* or *squadron* of aeroplanes would be temporarily or permanently attached to the independent cavalry, as in the case of the supports representing the other two Arms of the Service. Under these circumstances the command of the combined force would remain, as at present, with the cavalry leader.

[Just before the foregoing section ended, the book text is interrupted with a full-page photograph, Plate III, captioned: "R.A.F. Type B.E.2c. Fitted with R.A.F. (British Built) Engine. Compare Plate XIV" The Editors]

[Section 10 is noted as having been published originally on September 18th, 1914]

§10. *The Aeroplane as an Auxiliary to Tactical Operations.* The aeroplane in its employment in connection with tactical operations finds itself under conditions entirely different from those discussed in the preceding section, its duties are of a more varied character, and involve flying at lower altitudes than are compatible with security. It is likely to be almost continuously under fire, and, according to some of the experiences of the present war, it has almost as much to fear in this respect from its friends as its foes. Whereas the strategic reconnaissance machine is able to perform all its most useful work at high altitude, and avoid as far as possible the attention of, or actual contact with, the enemy, and evade pursuit by flight; the tactical machine (acting under the divisional command), whether engaged in local reconnaissance or in locating or directing gun-fire, or in other duties, must be prepared at once to tackle the enemy, and, in brief, to interfere as much as possible with the hostile aeroplane service. Under certain circumstances the instructions will undoubtedly be to make the aircraft of the enemy the first objective.

It is more than probable that it is in connection with the varied duties which in the future must fall to the Fourth Arm in its tactical usage, that differentiation of type and specialisation will eventually become the most marked. At present practically no attempt in the direction of specialisation has taken place. It is true the different machines in service vary considerably, and those responsible for the construction and specification of Service aeroplanes have already begun to talk of "reconnaissance machines" and "fighting machines;" but the distinction is one that has scarcely yet penetrated to the field of operations. When all has been said, differentiation of type must, from the Service standpoint, be looked upon as an evil, only to be justified when, and to the extent that, service conditions prove it to be necessary. So far even the broad distinction between machines for strategic reconnaissance and for tactical operations has scarcely been drawn or received recognition. The military aeroplane of to-day is something like the frontiersman's knife—made for nothing in particular, used for everything in general.

For the purpose of directing artillery fire the experience of the present war has shown the aeroplane to be effective almost beyond the most sanguine expectation. For this purpose it appears to have established its utility beyond question. Its duties in this respect may be regarded as a special branch of local reconnaissance, its function being to locate the objective and signal its whereabouts to the gun batteries to which it is attached; further to report and correct inaccuracies of fire. The exact mode or modes of signalling adopted do not so far appear to have been definitely disclosed. Some reports give the aeroplane as turning sharply when over the enemy's position; according to other accounts a smoke bomb of some kind is let fall to indicate the position to be attacked; other reports, again, mention lights as being used. It appears that lamps of sufficient power to be visible in daylight are actually being employed by the German aircraft. Possibly all these methods are in use experimentally, or different kinds of signals may be used for different purposes, to indicate initially the position, and subsequently to give corrections, either as to direction or range. Whatever the methods employed may be (and the details do not much concern us at the moment), they seem to be quite effective, and, it may be presumed, very considerably increase the fighting value of the guns. More than this, the value of aeroplane work will be relatively greater the longer the range; in fact, it may in future be found possible to employ heavy artillery of long range under conditions where, without the help of the aeroplane, it would be comparatively useless. As an illustration, there is nothing today to prevent a long-range battery, well served by its aeroplanes, from effectively shelling an enemy without knowing in the least the character of its objective—i.e., whether an infantry force or position, a body of cavalry, or the enemy's guns. In the present war the aeroplane appears to have been utilised by the German army, as a matter of regular routine, as an auxiliary to the artillery in the manner indicated. It has been reported again and again that the appearance of an aeroplane overhead has been the immediate prelude to the

bursting of shrapnel, frequently the very first shell being so accurately placed as to indicate that the method of signalling, and, in fact, the whole performance, must have been well thought out and equally well rehearsed.

It is well understood that the determination of the distance of an aeroplane of *known size* with approximate accuracy is a matter of perfect simplicity. Thus, if the aeroplane be flying fairly overhead, or directly towards or away from the observer, and the span be a known dimension, then by measuring the optical angle presented by the span, the distance or range is given by simple proportion. For example, holding a foot-rule square in front of one at arm's length—approximately 20 in. from the eye—the span, known to be, say, 36 ft., subtends an angle represented by, say, 1/2 in. on the scale; the distance is $\frac{20 \times 36}{0.5} = 1440$ ft. Using such rough-and-ready "apparatus," the degree of accuracy to be expected is not great; however, the author has found it quite sufficient to determine the altitude of a machine to within 5 or 6 per cent of the truth. If for the observer's arm and foot-rule we substitute a low-powered telescope or binocular of, say, 2 or 3 diameters magnification, with micrometer cross-wires, with which to follow up the apparent reduction in span of a receding aeroplane, until some prearranged signal is given, the range could undoubtedly be determined easily within 2 or 3 per cent. At 1 mile distance this means a degree of accuracy represented by a maximum error of about 40 yards, or sufficient to enable shrapnel to be dropped right on the mark. Parenthetically, it may be pointed out that the same method will enable the range of a hostile aeroplane to be determined, provided the type be identified, and its leading dimensions are known; it also suggests the importance of not flying exactly towards or away from, or exactly broadside to, any position of the enemy guarded by counter-aircraft artillery; flying end on to the enemy is also to be deprecated on the ground of fixity of direction.

§11. *Attack by Gun Fire.* An aeroplane operating in a hostile country is liable to attack by rifle and machine-gun fire, also by shell-fire from special anti-aeroplane artillery. It has comparatively little to fear from field artillery owing to the want of handiness of the ordinary field gun. The "laying" of a field-piece is far too clumsy a business to permit of its effective use on so small and rapidly moving a target as presented by an aeroplane in flight, although it may be effective when used against a dirigible. With regard to rifle or machine-gun (small bore) fire, calculation shows that aircraft is absolutely safe at an altitude of somewhat over 7000 ft.; it is in that region that the top of the trajectory lies for vertical shooting.

The duties of a strategic scout on long-distance work would, without doubt, permit of flying at such a high altitude, and it may be added that, although absolute immunity is not reached at less than about 7000 ft., a solitary aeroplane can only present a very unprofitable target at far lower altitudes. In fact, it may be taken that at, say, 5000 ft. or 6000 ft., the amount of small-arm ammunition required to bring down an aeroplane would be enormous. Not only has the velocity become so reduced as to render a "hit" capable of but little mischief, but the time of flight of the bullet, rising vertically to this altitude, would be about 8 or 9 seconds and the distance moved by the aeroplane 1000 ft., more or less. Therefore, it would be necessary to fire into quite a different part of the heavens from that in which the aeroplane is seen, something akin to sighting into the Great Bear to hit the Pole Star. Beyond this the gyroscopic drift of a bullet fired vertically is nil, against some 30 ft. or 40 ft. under normal conditions;[1] also the error due to the earth's rotation is a matter of about 30 ft. westward, and cannot be allowed for without taking reference to the compass bearing. Taking all of these things into account, it is evident that for the infantryman or gunner not specially trained, the task of bringing down an aeroplane flying at high altitude is no light one, especially when we recall the fact that for every inclination and bearing of the line of sight, the conditions differ. In designing the mounting of aeroplane-stopping artillery or machine-guns, it would be possible to render the sighting corrections for such items as gyroscopic drift and earth's rotation automatic; this could be done without difficulty, and would mean the elimination of errors whose combined value may amount to something like 60 ft. at 6000 ft. altitude—i.e., an angu-

lar magnitude represented roughly by the apparent diameter of the sun or moon.

The height to which aircraft artillery will carry is by no means subject to the same limitation as that of the small-bore machine-gun or rifle, the resistance of the air being many times greater than that due to gravity. Thus the ordinary rifle bullet, at 2,000 foot-seconds muzzle velocity, would carry to a height of over 60,000 ft. in *vacuo*, instead of approximately 7,000 ft. actual. If we take the case of a 1-pounder having the same velocity, its effective vertical range is well over 12,000 ft., and from that calibre upwards the range will, in practice, be more a question of the shell being properly directed than whether it will attain the height. At the best, firing from the ground at an aeroplane at high altitude, will required skilful gunnery, and when near the limit of the trajectory nothing but sheer good luck will render a hit effective. The angle of "lead" it is necessary to give to allow for the velocity of flight, as already stated, is one of the difficulties of high-altitude shooting. This angle is only constant so long as the velocity of the projectile is constant, assuming (as fairly represents the conditions) the flight speed not to vary; at extreme heights the velocity of the projectile has fallen so low that a very slight error in range-finding will be fatal to accuracy. The solution of this difficulty may be found in the employment of guns of about 3-in. bore i.e., a 12-pounder or 15-pounder, with the concurrent advantage of a full shrapnel charge, and, in shot-gun terminology, a larger killing circle. The obvious disadvantage, however, of artillery, in place of a light automatic or machine gun, lie in its want of portability and its unhandiness, difficulties which may, in course of time, be overcome.

All things considered, it would appear probably that attack on aeroplanes at high altitude from the ground will be found impracticable, or at least uncommercial. Not only have we to reckon with the various considerations above discussed, but also with the fact that, in our climate at least, not more than one day in four is sufficiently clear to render high-altitude shooting possible, and though it is true that an aeroplane, to make observation, cannot remain above or in the clouds, it presents but a poor mark under bad conditions.

An aeroplane operating at high altitude will probably need to be hunted and driven off or destroyed by armed machines of its own kind.

§12. *Defence from Gun-Fire.* It is manifestly not possible for an aeroplane to perform all the duties required of it, in connection with tactical operations at high altitude,[2] and whenever it descends below 5,000 ft., or whereabouts, it is liable to attack from beneath; in fact, at such moderate altitudes it must be considered as being under fire—mainly from machine-gun and rifle—the whole time it is over or within range of the enemy's lines. Protection from the rifle bullet may be obtained in either of two ways; the most vital portions of the machine, including the motor, the pilot, and gunner, can only be effectively protected by armour-plate; the remainder of the machine, including the wing members, the tail members, and portions of the fuselage not protected by armour, also the controls, struts, and the propeller, can be so constructed as to be *transparent* to rifle fire—that is to say, all these parts should be so designed that bullets will pass through without doing more than local injury and without serious effect on the strength of flying power of the machine as a whole; in certain cases components will require to be duplicated in order to realise this intention. It is important to understand clearly that any intermediate course is fatal. Either the bullet must be definitely resisted and stopped, or it must be let through with the least possible resistance; it is for the designer to decide in respect of each component which policy he will adopt. The thickness of the armour required will depend very much upon the minimum altitude at which, in the presence of the enemy, it is desired to fly, also upon the particular type of rifle and ammunition brought to bear. There is a great deal of difference in penetrative power, for example, between the round-nosed and pointed bullets used in an otherwise identical cartridge.

If it were not for the consideration of the weight of armour, there is no doubt that an altitude of about 1000 ft. would be found very well suited to most of the ordinary tactical duties of the aeroplane. At such an altitude, however, the

thickness of steel plate necessary becomes too serious an item for the present-day machine, even allowing for the very excellent and highly efficient bullet-proof treated steel which is now available; at the altitude in question the minimum thickness that will stop a 0.303 Mark VI. round-nose bullet is 3 mm. (1/8 in.), but if attacked by the modern pointed-nose Mauser, nothing short of 5 mm. or 6 mm. is of avail. If we compromise somewhat in the matter of altitude and prescribe 2000 ft. as the minimum height for which protection is to be given, the figures become 2 mm. (about 14 S.W. gauge) for the 0.303 round-nose bullet, and for the pointed Mauser 3 mm. or slightly over, at present it is not expected that it will pay to armour a machine for the duties in question more heavily; thus we may take 2000 ft. as representing the lower altitude limit of ordinary military flying.

[Just before the preceding sentence ends, the original text is interrupted by a full-page photograph, Plate IV, captioned: "SKELETON OF TYPE B.E.2. Showing position of Tank, Seats, Engine, and Body structure. Illustrates the extent of the vulnerable target presented by an Aeroplane. " The Editors]

Anything less than this will be referred to in the present series of articles as *low-altitude* flying. On this question of armour it cannot be too strongly insisted that anything less than the thickness necessary definitely to stop the projectile is worse than useless; a "mushroomed" bullet, possibly accompanied by a few detached fragments of steel, is infinitely more disagreeable and dangerous than a bullet which has not been upset.

An aeroplane armoured in all its vitals with 3 mm. steel, and otherwise designed on the lines indicated, flying at not less than 2000 ft. altitude, will be extremely difficult to bring down; so much so, that unless its exposed structural members be literally riddled and shattered by rifle and machine-gun, or unless a gun of larger calibre be brought to bear, it will be virtually impossible to effect its capture by gun-fire alone.

FOOTNOTES

[1] The normal sighting of a match rifle is arranged partially to correct for the gyroscopic drift.

[2] For military purposes we may take the term "high altitude" as defined by effective vertical range of small-arm fire—in other words, as denoting an altitude of some 5000 ft. or 6000 ft. or more.

US Army Logistics Management College Announces Course on Decision Analysis with Single and Multiple Objectives

Synopsis: Problems involving multiple criteria are of importance for decisionmaking in general, but especially significant in the military. There are many examples of decisions involving multiple criteria that must be made, including the selection of new weapon systems and other support systems, and the selection of alternative strategies. This course will provide a review of techniques for decisionmaking with multiple criteria, with an emphasis on understanding the strengths and weaknesses of alternative approaches that have been suggested for solving these problems.

Specific topics to be covered include the basic concepts of decision analysis, including the use of decision trees to provide a structure for analyzing risk and uncertainty, single attribute utility theory and multiple attribute utility theory for cases involving both risk and certainty, and techniques or problem solving with multiple decisionmakers.

Date:	19-23 June 1995
Place:	US Army Logistics Management College, Fort Lee, Virginia
Point of Contact:	The point of contact at AMSAA is Ms. Linda Stoflet, DSN 298-8966

Reminiscences of Early Operational Research

E.D.T. Strong, M.A., MSC

This is a record of the talk given "after dinner" to the 10th International Symposium on Military Operational Research held at the Royal Military College of Science, Shrivenham, UK, in 1993. The theme of the Symposium was "Operations Research and Future International Seacurity Operations." Mr. Strong was invited to speak as one of the very few UK analysts ever involved in OR in the field for a United Nations force. He subsequently filled a number of important posts in the Ministry of Defence, including those of Director Operational Analysis, Germany, Scientific Adviser to General Officer Commanding Northern Ireland, and Director of the Army Personnel Research Establishment, Farnborough, before retiring in 1986.

The North Korean Army invaded South Korea on the 25th of June 1950. In less than 3 days they had taken the capital, Seoul, 30 miles south of the border, and in a week they had pushed the South Koreans half-way to Pusan, their southernmost port, 200 miles away. The North Korean Army had some 400 T-34 tanks and a considerable number of tracked self-propelled guns. The armour of the South Korean Army consisted only of scout cars and half tracks, and the entire Army had only some 90 105mm howitzers so they were overwhelmed. Although taken by surprise (in fact he was on holiday), President Truman ordered American air and sea forces to give South Korean troops cover and support, and on 30th June he gave General MacArthur full authority to use all the troops under his command in Korea.

The American Eighth Army in Japan was an Army of Occupation that was undertrained, but it started to fly in troops as fast as possible. It first met the enemy on 6th July, but it was weeks before any significant force could be sent from the American mainland to Korea. The first US troops to arrive were equipped with the old 2.3 inch bazooka, which had been replaced by the 3.5 inch bazooka during WWII. This weapon was ineffective against the T34 tank. During the weeks of Summer 1950 the Americans and South Koreans barely managed to retain a defensive line that ended up on the banks of two rivers forming a perimeter round the port of Pusan. At that stage the North Koreans controlled 90 percent of the country. The North Korean aggression had been condemned by the Security Council of the United Nations, and many nations agreed to send troops to Korea. The first to arrive at the beginning of August was the British 27th Brigade from Hong Kong, which had, up to then, mainly been involved in border patrolling. Many of the UN contingents were slow to arrive, but by the summer of 1951 forces had come from Canada, Columbia, Australia, Belgium, France, Ethiopia, Great Britain, Greece, Holland, India, New Zealand, the Phillippines, Norway, Sweden, Thailand, Turkey, South Africa, and even Luxembourg, maybe 50,000 in all. By this time the South Koreans had about 350,000 men called up. The Americans had 230,000 in the country. Against them were 500,000 Chinese and 150,000 North Koreans. So it was a big war by any standards.

I haven't time to tell you about all the OR studies which were carried out. These included casualty analyses, armoured warfare studies, studies of the effectiveness of close air support and interdiction, studies on the effectiveness of artillery bombardment and naval gunfire support, and logistical studies which were very important because lack of roads, airfields and railway lines caused serious problems in the supply of ammunition and fuel. You will recognize that this was essentially World War II type of OR. What I propose to do instead is to give you some of my personal recollections of the war, and to indicate where OR was able to make a useful contribution, and perhaps where it wasn't.

I left Oxford University in 1949 and joined the Army Operational Research Group (AORG) at West Byfleet. The next year was really spent on training: a very useful course on Statistics under Professor Pearson at Imperial College, London, and a lot of on-the-job learning working at AORG with Eddie Benn mainly on tank studies, and with Geoff Wooldridge, who was concerned with the Army air and airborne forces. When the Korean war broke out, our War Office decided to send out a strong infantry brigade from UK with 3 infantry battalions, a tank regiment of our latest tanks, the Centurion Mark V, and plenty of artillery and engineer support. This was called 29 Brigade. I was somewhat surprised to be summoned to the Office of the Scientific Adviser to the Army Council, Tony Sargeaunt, in July 1950 and

asked whether I would like to join a small British Operational Research team going out with this Brigade. I went to see my Superintendent at AORG the next day, Neville Gadsby, who thought it would be good experience. But what about the danger of going to a war zone? One might become a casualaty or even get taken prisoner. It was important that one should be no worse off under these conditions than Army personnel.

Armed with these thoughts I went back to the Scientific Adviser, who was an experienced OR man, having been a member of the OR team that worked for 21st Army Group under General Montgomery in the Second World War. The result was that I found myself in uniform! I was told to report to the Middlesex Regiment at Mill Hill Barracks with a colleague from AORG, Reg Britain. We were given an indoctrination course on the organization, equipment, and customs of the British Army, which of course we didn't need. Then we were commissioned. I was immediately promoted to Captain. I was easily the youngest Captain in the British Army, and was given vast quantities of kit - battledress, greatcoat, boots, sleeping bag, campt kit, including mess tins and water bottles - so much indeed that I could hardly get in the car. In fact, I had to leave most of it behind because we were told we were going to fly out to Korea a few weeks later with the advance party of 29 Brigade.

We turned up at Lyneham to find we were going to fly out in a Hastings, a piston engined aircraft that could barely do 170 mph and had no night flying capability. This turned out to be a blessing in disguise, and we embarked upon what would have been considered in those days a very rare and expensive trip. The first night was spent on the ground in Malta, where we explored the delights of the Gut; the second in Beirut, which was in 1950 a peaceful and attractive city; the third in Habbaniyah where we could do more exploring in Baghdad. We then spent a night in Karachi, another in Negombo in Ceylon, which is a popular tourist center today, and arrived at Singapore with a spluttering engine. The pilot said he would go no further until it was replaced. As it had taken us 6 days to get out from England, we looked forward to a holiday in Singapore until another engine arrived. Unfortunately, we had not counted on the skill of the RAF fitters who removed the engine and repaired it in 48 hours. So we had to fly on to our destination, Sasebo in Japan, via Manila, arriving only 10 days after leaving England.

Crossing by ferry to Pusan, we went with the brigade advance party to 8th Army Headquarters who, to our surprise, knew a lot about operational research, largely because they had been briefed by Ellis Johnson, who was about to field a US Operations Research Office (ORO) team in Korea. They considered it would be a good idea if eventually we worked closely with this team, but in the meantime thought we would benefit from a little operational experience. I was attached to the US 1st Calvary Division, who had been there from the beginning, and to their 70th Tank Battalion, commanded by Colonel Bill Rogers, a very able tank commander. What they didn't tell us was that the Inchon landing was about to take place which changed the face of the war. This extremely bold landing at Inchon, the port of Seoul, required perfect timing as there was a 30-foot tide which when receding left mile-wide mud flats, so there was only a two-hour slot to get ashore.

The North Koreans were taken completely by surprise. The Marines captured the vital Kimpo airbase three days later, and Seoul was recaptured soon after. The North Koreans were caught in a trap between the Inchon invading Army in the North and the Eighth Army in the South and tried to extricate themselves as best they could, but 125,000 were taken prisoner. I found myself whisked off with the 70th Tank Battalion, chasing an enemy whose main object seemed to be to get out of South Korea. There was little fighting and little opportunity to do Operational Research. Our main problem was the wooden-boxed anti-tank mine, which was well used and concealed, difficult to detect, and with 20lb explosives could do a lot of damage. We crossed the 38th Parallel in the first week of October and ten days later were in Pyongyang, the capital of North Korea, where we stopped, largely because we had run out of supplies. However, complete victory now seemed to be

in sight. We all felt we would be home by Christmas, and I regretted that the opportunities to do worthwhile OR had been so few.

However, a tank battalion sitting in Pyongyang didn't know what was going on in the mind of the Theatre Commander. We were all quite astonished to be ordered to go North and close in on the Yalu River, the border with China. We felt there were too many unknown factors; the terrain, the weather, and especially the reaction of the Chinese. The terrain proved to be rugged, the weather uncertain, and this affected close air support. And in due course, on October 27th, the Chinese intervened with such devastating suddenness that many units were overrun before they knew what was happening. The Chinese had kept their movements well concealed, moving mostly at night, mostly on foot, avoiding roads in the daytime. Their units were self-sufficient, carrying simple food such as rice, beans and corn, and enough ammunition to last for a few days after which they would be relieved. At this stage they used no armour. They mainly used mortars for fire support, but they had plenty of infantry support weapons such as grenades, satchel charges and bazookas. Moving along paths and tracks, they surrounded the forward UN troops and set up a series of road blocks along the very few roads going South. The Chinese proved to be tough fighters who attacked without regard for casualties.

The 1st Cavalry Division could have held out if they had received ammunition and supplies, but their supply lines were cut and the roads were soon jammed with wrecked vehicles which made eventual withdrawal very difficult. When they did withdraw during the first week of November these roadblocks became the scene of heavy fighting.

The air forces, particulary the F-51 Mustangs which came in very low did a splendid job hitting the blocking Chinese troops on the slopes of the hills, particularly with napalm. Originally there were only 30 Mustangs left in Japan, but 145 came in an aircraft carrier in July 1950. I don't think any survived. At that time the F-80 Shooting Star was the main aircraft committed to Korea, but they were ill-suited to Korean conditions and no match for the MIG-15. It was not until the F-86 SabreJet arrived in December 1950 that we really had the edge. In the end, the F-86s had a 10:1 superiority over the MIGs in air combat.

The main hazard to the tank battalion I was travelling with was the Chinese infantry who tried to swarm over the tanks and immobilize them with satchel charges. The answer was to travel in groups of 4 and shoot the Chinese infantry off the decks of the tanks ahead. This worked as long as there was machine gun ammunition available, but as only 3000 rounds were carried in each tank there were problems. When the division pulled out, thankfully I was one of Col Roger's tank crew.

I was one of a small team who provided the after-action report, and 1st Calvary went into reserve. We found, for instance, that the 8th Calvary Regiment, which was the most forward and hence the last to pull out, lost in this one engagement half its strength and a lot of equipment including 10 tanks, 125 trucks, 12 105mm howitzers and at least a dozen recoilless riffles. One battalion, the 3rd, suffered particularly badly being left with only 10 officers and 200 men out of an initial strength of 800. We also did an after-action report on the 1st Marine Division who had been fighting on the Eastern side of the peninsula. They had suffered 4000 casualties, but when we visited them they had another 6000 men temporarily out of action with cold weather injuries due to lack of proper cold weather clothing. There was clothing in the theatre, but it didn't reach them in time.

I reckoned after this episode that I had enough operational experience and needed to get back to OR, which I hoped would be less hazardous. 29 Brigade had not arrived; they were coming out by sea and did not arrive until mid-November. So Reg Brittain and I joined ORO in a survey of what happened to North Korean armour in South Korea. Out of their force of some 400 tanks, virtually none survived. Our ground and air forces between them claimed to have destroyed over 1000, and we did find that most North Korean tanks had been hit more than once! It appeared that the American tanks with their 76mm and 90mm guns knocked out about 200 tanks for the loss of only 20. This was not surprising really because

in defence they were often in hull-down positions and advancing tanks found it difficult to get off the roads because of the terrain. The Air Force had flown about 25,000 sorties during this period, and many tanks had been hit by rockets or were burned out surrounded by napalm tanks. So the Air Force were probably responsible for most of the remainder. However, later on we were able to interrogate captured North Koreans tank crews, who were quite intelligent and definitely not country boys. They told us a significant proportion of tanks were abandoned because they had run out of fuel and ammunition, and they feared Air Force attack so much that they spent a lot of their time hiding in houses and barns so as to avoid being in the open.

As expected on November 25, the Chinese launched a major offensive with probably 300,000 men. The 8th Army took a severe pounding and withdrew southwards, in steady stages, in good order. They abandoned Pyongyang on 5 December and pulled back towards the 38th Parallel. The Chinese were only moving at about 6 miles per day; their supply lines were lengthening and they were receiving the full attention of our air power. However, as OR analysis showed, there is no such thing as closing off supply lines in a country as rugged as North Korea. The enemy were largely self-sufficient, able to carry their weapons and supplies on their backs. Moving at night or along foot trails by day they were often not visible from the air. This lesson was learned in Korea, but was forgotten by the time of the Vietnam War. The UN forces withdrew below the 38th Parallel on 15 December. A week later their Commander, General Walker, was killed in a jeep accident and General Ridgeway was appointed to succeed him.

General Ridgeway had hardly arrived when the Chinese started another offensive at the beginning of 1951 which led to the evacuation of Seoul and the withdrawal of UN forces to a defence force line (Han) south of the city. It was difficult to do OR at this time. Troops were on the move daily, their locations were uncertain, and I knew that contingency plans were being made for the evacuation of UN forces from Korea if that became necessary. In fact we did an OR study to determine how long this would take with the ships, planes, ports, airfields available. A lot of the studies we were asked to do concerned equipment performance, like the reliability of the Centurion tank, the effectiveness of its new stabilized gun control equipment, or the reliability of communications between ground forces and air support. This was not really OR; such questions could have easily been answered by the regiment's technical adjutants. Fortunately, the new defence line held and with the timely arrival of considerable reinforcements from the American mainland and the UN, in February, were able to make a surprise crossing of the Han River east of Seoul, and cut off the enemy's main supply route from North Korea. This was largely instrumental in forcing him to abandon the capital in mid-March, for the last time. The UN forces crossed the 38th parallel at the end of March, got pushed back by a new Chinese offensive in April (in the course of which our Air Forces flew over 7000 missions in one week) but returned there in May.

In June, the Soviet Deputy Foreign Minister, Jacob Malik, proposed a cease fire, and negotiations between the UN and the Communist forces opened in July. In fact the Armistice was not to be signed for another two years until July 1953, but there was now a different war situation. The enemy was too strong for us to drive him northwards. Equally he could not breach the UN lines which now had very heavy artillery support and increased air support due to new airfields coming into use. Key hills along the front changed hands several times, but there was, compared to the first year of the war, a comparative stalemate.

Both sides dug in as deep as they could, with sandbags and log-covered bunkers, minefields and barbed wire. The Chinese had extensive underground fortifications all along the front. They dug about 800 miles of tunnels and 3500 miles of trenches, which diminished the value of our air and artillery superiority. The fights for the hill positions, which proved good observation, were very intense. The artillery concentrations in such actions were quite formidable. In one action we monitored against feature Hill 916, the 38th Field Artillery fired 10,000 rounds between ten o'clock at night and four o'clock in the morning. There was no doubting the effec-

tiveness of this particular bombardment and the hill was taken. But whether it was cost-effective is another matter, as we lost the hill again some days later.

Another study I recall was on battlefield illumination, as the Chinese liked to attack at night. The outcome was a greatly improved system using not only the mortar and artillery flares that had always been used, but also powerful searchlights bouncing their glare off the clouds, and planes called Fireflies which dropped flares over the battlefield and were very flexible because they were not terrain limited. Another study we did was on the effectiveness of body armour. The Americans lightweight body armour (8 lbs) made up of layers of glass fibre and nylon reduced fatal chest and abdominal wounds by 70%, but it was hardly used by non-American troops.

We worked very closely with ORO, who had a dozen men in the theatre, often being involved in joint studies. ORO had set up an office in Tokyo for administrative purposes and to liase with GHQ which was in Tokyo. This proved a great advantage to us. We could get our reports typed there, and they had good worldwide communications which we did not have. Not least, there were some really nice American girls there, so there was great competition for liaison trips back to the Tokyo office. My colleague Reg Britain, who was senior to me, took unfair advantage of rank, to such an extent that he ended up by marrying one of the girls. But that is another story! In due course, we were joined by a two-man Canadian OR team and a single New Zealand OR man, and we had a third man join the UK team.

During one of our visits to the Thai battalion we were surprised to be asked if a young officer could join us to find out more about OR. He duly arrived, rather inadequately dressed as it was the middle of the winter; so cold in fact, that the rivers froze and I have a photograph of Centurion tanks crossing the ice over the Imjin River - and the Centurion weighed 50 tons. This unfortunate man had to share my tent in which I had a good stove. But he let it get red hot and burnt down the tent. I was very cross as tents were hard to come by. We had words and we lost a potential recruit to OR. I never saw him again.

Although ORO did the same work as we did in the field, in Tokyo, they were engaged in more wide-ranging studies of a strategic nature, but we were not involved in those. A lot of them involved studying the effectiveness of air attacks on North Korea, where, in 1952 particularly, the bombing campaign war stepped up. For instance, the US bombed the four most vital dams and power complexes in North Korea, including the Suprung Dam on the Yalu River which supplies 90 percent of North Korea's power supplies. The raid by some 500 planes knocked out the generators. North Korea had a power blackout for a fortnight and lost a lot of its electrical power for the rest of the war.

A series of large raids were carried out on towns, particularly Pyongyang. In the biggest one over 1000 sorties were flown, and there were 6000 deaths in a city whose population had dropped to 50,000. But despite these large efforts it was not possible to isolate the battlefield and the ground war went on much as before.

Working in the field has one big advantage in that you are tasked directly by whoever wants the job done. If you have queries, you can go and talk to him. Also when you have done the job, you can report back to your tasker to make sure you have covered the ground fully. Sometimes you even get the satisfaction of seeing your recommendations acted on quickly. However, in Korea, we felt we might have directed our efforts better had we known more about the broad picture and the directives which were given to the theatre commanders setting out the mission they were required to accomplish.

On one occasion the Scientific Adviser, Tony Sargeaunt, came to visit us in Korea, and we met a Korean Defence Minister, who had been at the same Oxford College as I had. He was a charming chap, and invited us to a party. Now in that part of the world parties are not held at home, but in a eating house with japanese geishas in attendance; in Korea they are called Kaesong girls. They are always beautifully dressed, and sing and dance and play party games with you and keep the evening buzzing along. We got quite friendly with the Minister,

and I was astonished when we met him next to find out how politically ignorant I was about what had been going on at higher levels. There was talk about the value of bombing the Yalu River bridges unless one also bombed the Manchurian airbases from which the MIGs defending them operated (only half were knocked out despite constant attacks), about the need for a second front which meant Chinese Nationalist forces invading from Formosa and being used in Korea to beef up our numbers, the possible use of A-weapons, first requested by General MacArthur in December 1951, and similar themes, even the possibility of a Russian invasion of Japan. It certainly opened my eyes and made me realize although I was satisfied with my own work, it was very local and did not begin to impinge on issues of this magnitude.

Of course, OR operated at all levels, as it has always done, briefing Commanders at high level and carrying out field studies at a much lower level. However, I felt that our tasking in Korea didn't involve much forward thinking. It was concerned with the problems of the day, and sometimes, by the time the results were obtained the scenario had changed; for instance, from a mobile war in the first year to a static war thereafter. So some of our reports were of historical value only.

In setting up our OR section, no thought was given to support; there were no drivers, no clerks or, indeed, anybody to help out. But for the help given us by ORO, we would have found it difficult to operate. One problem was theft. You could not leave your vehicle or tent anywhere except in guarded compounds, and that was not generally possible. I got over this by recruiting a Korean lad, about 15, whose sole job was to guard my property, which he did very well. He started out by being rather thin and scrawny, but by the time I left he had filled out even on our rations; he eventually went into the South Korean Army where I gather he has done well. If I had to go on a similar posting again, I would insist on a driver, who could double as a guard. ORO were not in uniform, although they wore military style clothing with Operational Analysis shoulder badges. I don't think they needed to be in uniform, particularly as they had a first-rate Military Executive Officer, Lt Col Charles Billingsley, who liasised with the military and smoothed the way for them. I found the American units the easiest to deal with. They didn't resent scientists observing them and asking them questions. Some other nationalities were very suspicious of us, to the extent of being uncooperative. Even when the 1st British Commonwealth Division was formed with British Canadian, Australian and New Zealand troops, most commanding officers were very helpful, but there were some who clearly didn't want you on their patch. On these occasions I would have been a lot better off as a civilian. It is no good trying to argue with a superior officer if you are in uniform.

What other lessons can be learnt about OR from this episode in Korea? Of ourse, it is obvious that OR methodology and models have improved out of all recognition since 1950, so the present day approach would be different. We would be looking to see where our knowledge could be applied rather than waiting to be tasked. However, it is difficult to predict where UN intervention will take place and in what form, so if OR is to play a part, trained analysts must be available at relatively short notice and must have thought about likely problems in advance.

Then there is the problem of tasking. Commanders of UN operations must know what OR is capable of if they are to benefit from the presence of OR teams. (There must be empathy between the people responsible for operations and their advisors.) This takes time. I cannot see a permanent OR team being set up at the United Nations in peacetime, however desirable that might be. So I believe that at the outset of conflict, OR teams will work with their own national contingents. Hopefully as in Korea they will cooperate closely with each other and even become integrated.

Looking back one can see that this was a war neither side won. Nobody knows what the Chinese and North Korean casualties were, certainly over a million. US had 158,000 casualties with 54,000 killed and the UN forces had 17,000 casualties of which 5,000 were British. The biggest winner was probably Chiang Kai Shek, as the Korean war saved him having to fight the

Chinese and suffer the devastation that Korea experienced. One thing I was sorry about is that OR had no part to play in the long peace negotiations that ended the war. Some years later, when the Mutual and Balanced Force Reduction talks were taking place in Vienna, the proposals made by the Soviets were often evaluated overnight using theatre models. Of course, in Korea we did not have computer models available, but certainly in Tokyo an analytical or war gaming approach could have been available and might have been a useful evaluation tool. At least we could have known what was happening in Panmunjon.

One of the saddest things after the Korean War was that the official position of the US Defense Department and our own Ministry of Defence was that they felt that no real changes in doctrine had emerged from the War and that relatively few items could be described as useful lessons for future campaigns. This was definitely not the view of the OR community who felt that we had a lot to learn about fighting local wars in Asia or the Far East. But we were not able to bring sufficient influence to bear on the military command to change their way of thinking and I think we paid for this in later years. Had Ellis Johnson, who had a lot of influence, stayed in Far East Command things would have been different. So the lesson from this is that you need a respected leader able to take part in planning and discussions at the top theatre level if OR is to play its full part in any campaign.

Readers interested in similar historical accounts are referred to J. Opl Res Soc Vol 40, No 2, pp 115-136, 1989; J. Opl Res Soc Vol 43, No 6, pp 569-577, 1992; and to Operations Research Vol 40, No 4, pp 663-639, 1992.

American Military University Receives National Accreditation

American Military University (AMU), a distance-education graduate school, with headquarters in Manassas Park, Virginia, received accreditation effective 6 January 1995 from the Distance Education and Training Council (DETC). Founded in 1926, DETC is recognized by the U.S. Department of Education as a national accreditation agency, operating under Public Law, for distance education schools offering certificates and academic degrees up to and including the Master's degree. The State Council on Higher Education for Virginia authorized AMU to issue the Master's of Arts in Military Studies degree, effective 21 February 1995, completing the accrediting and licensure process.

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